Technical Appendix: Economic Uncertainty, Disagreement, and Credit Markets

We want to compute the following moment-generating function

$$F\left(A,\eta,m_A^1,\Psi_A,\Psi_z,t,u;\epsilon,\chi\right)=E_{A,\eta,m_A^1,\Psi_A,\Psi_z}\left(A(u)^{\epsilon}\eta(u)^{\chi}\right).$$

This function satisfies the following partial differential equation (PDE):

$$0\equiv\mathcal{D}F\left(A,\eta,m_A^1,\Psi_A,\Psi_z,t,u;\epsilon,\chi\right)+\frac{\partial F}{\partial t}\left(A,\eta,m_A^1,\Psi_A,\Psi_z,t,u;\epsilon,\chi\right),\tag{1}$$

with the initial condition $F\left(A,\eta,m_A^1,\Psi_A,\Psi_z,t,t;\epsilon,\chi\right)=A^{\epsilon}\eta^{\chi}$, and where $\mathcal{D}$ is the differential generator of the multivariate process $\left(A(t),\eta(t),m_A^1(t),\Psi_A(t),\Psi_z(t)\right)$ under the probability measure of agent 1. Spelling out Feynman-Kač (2), we get

$$\begin{aligned}
0 \;=\;& \frac{\partial F}{\partial A}Am_A^1+\frac{\partial F}{\partial m_A^1}\left(a_{0A}+a_{1A}m_A^1\right)+\frac{\partial F}{\partial \Psi_A}\left(\left(a_{1A}+\frac{\gamma_A^2}{\sigma_A^2}\right)\Psi_A+\left(\frac{\alpha\gamma_A^2+\beta\gamma_{Az}^2}{\sigma_A\sigma_z}\right)\Psi_z\right)\\
&+\frac{\partial F}{\partial \Psi_z}\left(\left(a_{1z}+\frac{\alpha\gamma_{Az}^2+\beta\gamma_z^2}{\sigma_z^2}\right)\Psi_z+\frac{\gamma_{Az}^2}{\sigma_A\sigma_z}\Psi_A\right)+\frac{1}{2}\frac{\partial^2 F}{\partial A^2}\left(A\sigma_A\right)^2+\frac{1}{2}\frac{\partial^2 F}{\partial\left(m_A^1\right)^2}\left(\left(\frac{\gamma_A^1}{\sigma_A}\right)^2+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)^2\right)\\
&+\frac{1}{2}\frac{\partial^2 F}{\left(\partial\Psi_A\right)^2}\left(\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)^2+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)^2\right)\\
&+\frac{1}{2}\frac{\partial^2 F}{\left(\partial\Psi_z\right)^2}\left(\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)^2+\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)^2\right)+\frac{1}{2}\frac{\partial^2 F}{\left(\partial\eta\right)^2}\eta^2\left(\Psi_A^2+\left(\alpha\frac{\sigma_A}{\sigma_z}\Psi_A+\beta\Psi_z\right)^2\right)\\
&+\frac{\partial^2 F}{\partial A\partial m_A^1}\gamma_A^1 A+\frac{\partial^2 F}{\partial A\partial\Psi_A}\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A}\right)A+\frac{\partial^2 F}{\partial A\partial\Psi_z}\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_z}\right)A-\frac{\partial^2 F}{\partial A\partial\eta}A\eta\Psi_A\sigma_A\\
&+\frac{\partial^2 F}{\partial m_A^1\partial\Psi_A}\left(\frac{\gamma_A^1\left(\gamma_A^1-\gamma_A^2\right)}{\sigma_A^3}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right)\\
&+\frac{\partial^2 F}{\partial m_A^1\partial\Psi_z}\left(\frac{\gamma_A^1\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A^2\sigma_z}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right)\\
&-\frac{\partial^2 F}{\partial m_A^1\partial\eta}\eta\left(\Psi_A\frac{\gamma_A^1}{\sigma_A}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&-\frac{\partial^2 F}{\partial\Psi_A\partial\eta}\eta\left(\Psi_A\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&-\frac{\partial^2 F}{\partial\Psi_z\partial\eta}\eta\left(\Psi_A\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_z\sigma_A}+\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&+\frac{\partial^2 F}{\partial\Psi_A\partial\Psi_z}\left(\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right)+\frac{\partial F}{\partial t}.
\end{aligned}$$

The solution to this PDE takes the functional form

$$F(A,\eta,m_A^1,\Psi_A,\Psi_z,t,u;\epsilon,\chi)=A^{\epsilon}\eta^{\chi}F_{m_A^1}\left(m_A^1,t,u;\epsilon\right)F_{\Psi_A,\Psi_z}\left(\Psi_A,\Psi_z,t,u;\epsilon,\chi\right)=:A^{\epsilon}\eta^{\chi}\tilde{F}\left(m_A^1,\Psi_A,\Psi_z,t,u;,\epsilon,\chi\right).$$

Plugging in this expression into equation (2), yields:

$$
\begin{aligned}
0 \quad = \quad & \tilde{F}\epsilon m_A^1 + \frac{\partial \tilde{F}}{\partial m_A^1}\left(a_{0A} + a_{1A} m_A^1\right) + \frac{\partial \tilde{F}}{\partial \Psi_A}\left(\left(a_{1A} + \frac{\gamma_A^2}{\sigma_A^2}\right)\Psi_A + \left(\frac{\alpha\gamma_A^2 + \beta\gamma_{Az}^2}{\sigma_A\sigma_z}\right)\Psi_z\right) \\
& + \frac{\partial \tilde{F}}{\partial \Psi_z}\left(\left(a_{1z} + \frac{\alpha\gamma_{Az}^2 + \beta\gamma_z^2}{\sigma_z^2}\right)\Psi_z + \frac{\gamma_{Az}^2}{\sigma_A\sigma_z}\Psi_A\right) + \frac{1}{2}\epsilon(\epsilon-1)\tilde{F}\sigma_A^2 + \frac{1}{2}\frac{\partial^2 \tilde{F}}{\partial\left(m_A^1\right)^2}\left(\left(\frac{\gamma_A^1}{\sigma_A}\right)^2 + \left(\frac{\alpha\gamma_A^1 + \beta\gamma_{Az}^1}{\sigma_z}\right)^2\right) \\
& + \frac{1}{2}\frac{\partial^2 \tilde{F}}{\left(\partial \Psi_A\right)^2}\left(\left(\frac{\gamma_A^1 - \gamma_A^2}{\sigma_A^2}\right)^2 + \left(\frac{\alpha\left(\gamma_A^1 - \gamma_A^2\right) + \beta\left(\gamma_{Az}^1 - \gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)^2\right) \\
& + \frac{1}{2}\frac{\partial^2 \tilde{F}}{\left(\partial \Psi_z\right)^2}\left(\left(\frac{\gamma_{Az}^1 - \gamma_{Az}^2}{\sigma_A\sigma_z}\right)^2 + \left(\frac{\alpha\left(\gamma_{Az}^1 - \gamma_{Az}^2\right) + \beta\left(\gamma_z^1 - \gamma_z^2\right)}{\sigma_z^2}\right)^2\right) + \frac{1}{2}\chi(\chi-1)\tilde{F}\left(\Psi_A^2 + \left(\alpha\frac{\sigma_A}{\sigma_z}\Psi_A + \beta\Psi_z\right)^2\right) \\
& + \frac{\partial \tilde{F}}{\partial m_A^1}\epsilon\gamma_A^1 + \frac{\partial \tilde{F}}{\partial \Psi_A}\left(\frac{\gamma_A^1 - \gamma_A^2}{\sigma_A}\right)\epsilon + \frac{\partial \tilde{F}}{\partial \Psi_z}\left(\frac{\gamma_{Az}^1 - \gamma_{Az}^2}{\sigma_z}\right)\epsilon - \epsilon\chi\tilde{F}\Psi_A\sigma_A \\
& + \frac{\partial^2 \tilde{F}}{\partial m_A^1 \partial \Psi_A}\left(\frac{\gamma_A^1\left(\gamma_A^1 - \gamma_A^2\right)}{\sigma_A^3} + \left(\frac{\alpha\gamma_A^1 + \beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_A^1 - \gamma_A^2\right) + \beta\left(\gamma_{Az}^1 - \gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right) \\
& + \frac{\partial^2 \tilde{F}}{\partial m_A^1 \partial \Psi_z}\left(\frac{\gamma_A^1\left(\gamma_{Az}^1 - \gamma_{Az}^2\right)}{\sigma_A^2\sigma_z} + \left(\frac{\alpha\gamma_A^1 + \beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1 - \gamma_{Az}^2\right) + \beta\left(\gamma_z^1 - \gamma_z^2\right)}{\sigma_z^2}\right)\right) \\
& - \frac{\partial \tilde{F}}{\partial m_A^1}\chi\left(\Psi_A\frac{\gamma_A^1}{\sigma_A} + \left(\frac{\alpha\gamma_A^1 + \beta\gamma_{Az}^1}{\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z} + \beta\Psi_z\right)\right) \\
& - \frac{\partial \tilde{F}}{\partial \Psi_A}\chi\left(\Psi_A\frac{\gamma_A^1 - \gamma_A^2}{\sigma_A^2} + \left(\frac{\alpha\left(\gamma_A^1 - \gamma_A^2\right) + \beta\left(\gamma_{Az}^1 - \gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z} + \beta\Psi_z\right)\right) \\
& - \frac{\partial \tilde{F}}{\partial \Psi_z}\chi\left(\Psi_A\frac{\gamma_{Az}^1 - \gamma_{Az}^2}{\sigma_z\sigma_A} + \left(\frac{\alpha\left(\gamma_{Az}^1 - \gamma_{Az}^2\right) + \beta\left(\gamma_z^1 - \gamma_z^2\right)}{\sigma_z^2}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z} + \beta\Psi_z\right)\right) \\
& + \frac{\partial^2 \tilde{F}}{\partial \Psi_A \partial \Psi_z}\left(\left(\frac{\gamma_A^1 - \gamma_A^2}{\sigma_A^2}\right)\left(\frac{\gamma_{Az}^1 - \gamma_{Az}^2}{\sigma_A\sigma_z}\right) + \left(\frac{\alpha\left(\gamma_A^1 - \gamma_A^2\right) + \beta\left(\gamma_{Az}^1 - \gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1 - \gamma_{Az}^2\right) + \beta\left(\gamma_z^1 - \gamma_z^2\right)}{\sigma_z^2}\right)\right) + \frac{\partial \tilde{F}}{\partial t}.
\end{aligned}
$$

We can first factor out the expressions that do not involve η and Ψ (given in blue) and solve for $F_{m_A^1}$ by direct integration. To this end, we guess the following functional form:

$$F_{m_A^1}(m_A^1, t, u, \epsilon) = \exp(A(\epsilon, u-t)m_A^1 + C(\epsilon, u-t)),$$

with the explicit solutions for $A(\epsilon, \tau)$ and $C(\epsilon, \tau)$ given by:

$$
\begin{aligned}
A(\epsilon, u-t) \quad &= \quad \frac{\epsilon\left(e^{a_{1A}(u-t)} - 1\right)}{a_{1A}}, \\
C(\epsilon, u-t) \quad &= \quad \frac{1}{2}\epsilon\left(\epsilon - 1\right)\sigma_A\left(u-t\right) + \frac{1}{a_{1A}}\left(a_{0A} + \epsilon\gamma_A^1\right)\left(e^{-a_{1A}(u-t)} + u - t\right) \\
& \qquad + \frac{1}{a_{1A}}\left(\left(\frac{\gamma_A^1}{\sigma_A}\right)^2 + \left(\frac{\alpha\gamma_A^1 + \beta\gamma_{Az}^1}{\sigma_z}\right)^2\right)\left(\frac{3}{2}e^{a_{1A}(u-t)} - a_{1A}\left(u-t\right)\right).
\end{aligned}
$$

Next, we guess the following functional form:

$$F_{\Psi_A,\Psi_z}(\Psi_A,\Psi_z,t,\epsilon,\chi,u) = \exp\Big(A_0(\epsilon,\chi,u-t)+B_1(\epsilon,\chi,u-t)\Psi_A+B_2(\epsilon,\chi,u-t)\Psi_z$$
$$+C_1(\epsilon,\chi,u-t)\Psi_A^2+C_2(\epsilon,\chi,u-t)\Psi_z^2+D_0(\epsilon,\chi,u-t)\Psi_A(t)\Psi_z\Big).$$

From this guess, we obtain the derivatives:

$$\frac{\partial\tilde{F}}{\partial\Psi_A} = \tilde{F}\left(B_1(u-t)+2C_1(u-t)\Psi_A+D_0(u-t)\Psi_z\right),$$
$$\frac{\partial^2\tilde{F}}{\partial\Psi_A^2} = \tilde{F}\left(\left(B_1(u-t)+2C_1(u-t)\Psi_A+D_0(u-t)\Psi_z\right)^2+2C_1(u-t)\right),$$
$$\frac{\partial\tilde{F}}{\partial\Psi_z} = \tilde{F}\left(B_2(u-t)+2C_2(u-t)\Psi_z+D_0(u-t)\Psi_A\right),$$
$$\frac{\partial^2\tilde{F}}{\partial\Psi_z^2} = \tilde{F}\left(\left(B_2(u-t)+2C_2(u-t)\Psi_z+D_0(u-t)\Psi_A\right)^2+2C_2(u-t)\right),$$
$$\frac{\partial\tilde{F}}{\partial\Psi_A\Psi_z} = \tilde{F}\left(\left(B_1(u-t)+2C_1(u-t)\Psi_A+D_0(u-t)\Psi_z\right)\left(B_2(u-t)+2C_2(u-t)\Psi_z+D_0(u-t)\Psi_A\right)+D_0(u-t)\right),$$
$$\frac{\partial\tilde{F}}{\partial t} = -\tilde{F}\left(A_0'(u-t)+B_1'(u-t)\Psi_A+B_2'(u-t)\Psi_z+C_1'(u-t)\Psi_A^2+C_2'(u-t)\Psi_z^2+D_0'(u-t)\Psi_A\Psi_z\right),$$

which, plugged–in into the initial differential equation imply:

$$
\begin{aligned}
0 \;=\;& (B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)\left(\left(a_{1A}+\frac{\gamma_A^2}{\sigma_A^2}\right)\Psi_A+\left(\frac{\alpha\gamma_A^2+\beta\gamma_{Az}^2}{\sigma_A\sigma_z}\right)\Psi_z\right)\\
&+(B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)\left(\left(a_{1z}+\frac{\alpha\gamma_{Az}^2+\beta\gamma_z^2}{\sigma_z^2}\right)\Psi_z+\frac{\gamma_{Az}^2}{\sigma_A\sigma_z}\Psi_A\right)+\\
&+\frac{1}{2}\left((B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)^2+2C_1(\tau)\right)\left(\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)^2+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)^2\right)\\
&+\frac{1}{2}\left((B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)^2+2C_2(\tau)\right)\left(\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)^2+\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)^2\right)\\
&+\frac{1}{2}\chi(\chi-1)\left(\Psi_A^2+\left(\alpha\frac{\sigma_A}{\sigma_z}\Psi_A+\beta\Psi_z\right)^2\right)\\
&+(B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A}\right)\epsilon+(B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_z}\right)\epsilon-\epsilon\chi\Psi_A\sigma_A\\
&+A(\epsilon,\tau)\,(B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)\left(\frac{\gamma_A^1\left(\gamma_A^1-\gamma_A^2\right)}{\sigma_A^3}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right)\\
&+A(\epsilon,\tau)\,(B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)\left(\frac{\gamma_A^1\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A^2\sigma_z}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right)\\
&-A(\epsilon,\tau)\chi\left(\Psi_A\frac{\gamma_A^1}{\sigma_A}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&-(B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)\,\chi\left(\Psi_A\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&-((B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)\,\chi\left(\Psi_A\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_z\sigma_A}+\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\left(\alpha\Psi_A\frac{\sigma_A}{\sigma_z}+\beta\Psi_z\right)\right)\\
&+((B_1(\tau)+2C_1(\tau)\Psi_A+D_0(\tau)\Psi_z)\,(B_2(\tau)+2C_2(\tau)\Psi_z+D_0(\tau)\Psi_A)+D_0(\tau))\\
&\times\left(\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right)\\
&-\left(A_0'(\tau)+B_1'(\tau)\Psi_A+B_2'(\tau)\Psi_z+C_1'(\tau)\Psi_A^2+C_2'(\tau)\Psi_z^2+D_0'(\tau)\Psi_A\Psi_z\right).
\end{aligned}
$$

It follows that functions A_0, B_1, B_2, C_1, C_2 and D_0 must solve the following system of ODEs

$$
\begin{aligned}
C_1'(\tau) &= 2a_1 C_1^2(\tau) + 2b_1 C_1(\tau) + \frac{1}{2}\tilde{a}_1 D_0^2(\tau) + \tilde{b}_1 D_0(\tau) + 2c_1 C_1(\tau) D_0(\tau) + d_1, && (2)\\
C_2'(\tau) &= 2\tilde{a}_1 C_2^2(\tau) + 2b_2 C_2(\tau) + \frac{1}{2}a_1 D_0^2(\tau) + \tilde{b}_2 D_0(\tau) + 2c_1 C_2(\tau) D_0(\tau) + d_2, && (3)\\
D_0'(\tau) &= c_1 D_0^2(\tau) + \frac{1}{2}\left(b_1 + b_2\right) D_0(\tau) + 2\tilde{b}_1 C_2(\tau) + 2\tilde{b}_2 C_1(\tau) + 2a_1 C_1(\tau) D_0(\tau) + 2\tilde{a}_1 C_2(\tau) D_0(\tau) \\
&\quad + 4c_1 C_1(\tau) C_2(\tau) + d_5, && (4)\\
B_1'(\tau) &= b_1 B_1(\tau) + \tilde{b}_1 B_2(\tau) + a_1 B_1(\tau) C_1(\tau) + \tilde{a}_1 B_2 D_0(\tau) + c_1 B_1(\tau) D_0(\tau) + 2c_1 C_1(\tau) B_2(\tau) \\
&\quad + d_3 - e^{a_{1A}\tau}\tilde{d}_3 + C_1(\tau)\left(\tilde{b}_3 e^{a_{1A}\tau} - b_3\right) + D_0(\tau)\left(\tilde{c}_3 e^{a_{1A}\tau} - c_3\right) + d_3, && (5)\\
B_2'(\tau) &= b_2 B_2(\tau) + b_1 B_1(\tau) + a_1 B_1(\tau) D_0(\tau) + \tilde{a}_1 B_2 C_2(\tau) + \left(2B_1(\tau) C_2(\tau) + B_2(\tau) D_0(\tau)\right) c_1 \\
&\quad + C_2(\tau)\left(\tilde{b}_3 e^{a_{1A}\tau} - b_3\right) + D_0(\tau)\left(\tilde{c}_3 e^{a_{1A}\tau} - c_3\right) + d_4\left(e^{a_{1A}\tau} - 1\right), && (6)\\
A_0'(\tau) &= \frac{1}{2}a_1 B_1^2(\tau) + \frac{1}{2}\tilde{a}_1 B_2^2(\tau) + \tilde{a}_1 C_2(\tau) + b_3 B_1(\tau) + \frac{1}{2}c_3 B_2(\tau) + c_1 B_1(\tau) B_2(\tau), && (7)
\end{aligned}
$$

subject to the initial condition:

$$C_1(0) = C_2(0) = B_1(0) = B_2(0) = D_0(0) = A_0(0) = 0.$$

In these equations, the coefficients are given explicitly by:

$$
\begin{aligned}
a_1 &= \left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)^2+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)^2,\\
b_1 &= a_{1A}+\frac{\gamma_A^2}{\sigma_A^2}-\chi\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A}+\frac{\alpha\sigma_A}{\sigma_z}\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right),\\
\tilde{a}_1 &= \left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)^2+\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)^2,\\
\tilde{b}_1 &= \frac{\gamma_{Az}^2}{\sigma_A\sigma_z}-\chi\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}+\left(\frac{\alpha\sigma_A}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right),\\
c_1 &= \left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A^2}\right)\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_A\sigma_z}\right)+\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right),\\
d_1 &= \frac{1}{2}\chi\left(\chi-1\right)\left(1+\left(\frac{\alpha\sigma_A}{\sigma_z}\right)^2\right),\\
b_2 &= a_{1z}+\frac{\alpha\gamma_{Az}^2-\chi\beta\gamma_{Az}^2}{\sigma_z^2}+\beta\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right),\\
\tilde{b}_2 &= \frac{\alpha\gamma_A^2+\beta\gamma_{Az}^2}{\sigma_A\sigma_z}-\chi\beta\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right),\\
d_2 &= \frac{1}{2}\chi\left(\chi-1\right)\beta^2,\\
b_3 &= \epsilon\left(\frac{\gamma_A^1-\gamma_A^2}{\sigma_A}\right)-\frac{\epsilon}{a_{1A}}\left(\frac{\gamma_A^1\left(\gamma_A^1-\gamma_A^2\right)}{\gamma_A^3}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right),\\
\tilde{b}_3 &= \frac{\epsilon}{a_{1A}}\left(\frac{\gamma_A^1\left(\gamma_A^1-\gamma_A^2\right)}{\sigma_A^3}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_A^1-\gamma_A^2\right)+\beta\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A\sigma_z}\right)\right),\\
c_3 &= 2\epsilon\left(\frac{\gamma_{Az}^1-\gamma_{Az}^2}{\sigma_z}\right)-\frac{2\epsilon}{a_{1A}}\left(\frac{\gamma_A^1\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A^2\sigma_z}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z^2}\right)\right),\\
\tilde{c}_3 &= \frac{2\epsilon}{a_{1A}}\left(\frac{\gamma_A^1\left(\gamma_{Az}^1-\gamma_{Az}^2\right)}{\sigma_A^2\sigma_z}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\left(\gamma_{Az}^1-\gamma_{Az}^2\right)+\beta\left(\gamma_z^1-\gamma_z^2\right)}{\sigma_z}\right)\right),\\
d_3 &= \epsilon\chi\left(\sigma_A-\frac{1}{a_{1A}}\left(\frac{\gamma_A^1}{\sigma_A}+\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\sigma_A}{\sigma_z}\right)\right),\\
\tilde{d}_3 &= -\frac{\epsilon\chi}{a_{1a}}\left(\frac{\gamma_A^1}{\sigma_A}+\left(\frac{\alpha\gamma_A^1+\beta\gamma_{Az}^1}{\sigma_z}\right)\left(\frac{\alpha\sigma_A}{\sigma_z}\right)\right),\\
d_5 &= \frac{\alpha\sigma_A}{\sigma_z}\chi\left(\chi-1\right).
\end{aligned}
$$

To solve the system of equations (2)-(7), we first solve equations (2)-(4). First, we observe that the system of differential equations (2)-(4) can be written as the following matrix Riccati equation:

$$\frac{dA}{d\tau}=AM'MA+AP+P'A+D \tag{8}$$

with coefficient matrices defined by:

$$A = \begin{pmatrix} C_1 & D_0 \\ D_0 & C_2 \end{pmatrix}, \quad M = \begin{pmatrix} 2\sqrt{a_1} & 0 \\ \frac{c_1}{\sqrt{a_1}} & \sqrt{2\tilde{a}_1 - \frac{c_1^2}{a_1}} \end{pmatrix}, \quad P = \begin{pmatrix} 2b_1 & \tilde{b}_2 \\ \tilde{b}_1 & 2b_2 \end{pmatrix} \quad D = \begin{pmatrix} d_1 & d_5 \\ d_5 & d_1 \end{pmatrix}.$$

The matrix Riccati equation (8) can be solved in closed form by transforming it into a locally equivalent linear system of ordinary differential equations by a homogenization procedure (Radon's Lemma). Let

$$\begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix} = \exp\left(t \begin{pmatrix} P & M'M \\ D & -P' \end{pmatrix}\right).$$

Then, the solution of differential equation (8) is:

$$A(t) = C_{22}(t)^{-1} C_{21}(t), \tag{9}$$

using the fact that $A(0) = 0$. The solutions for B_1, B_2, and A_0 follow by direct integration. This concludes the proof.

□