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Habit Formation and Macroeconomic Models of the Term Structure of Interest Rates

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ABSTRACT

This paper introduces a new class of nonaffine models of the term structure of interest rates that is supported by an economy with habit formation. Distinguishing features of the model are that the interest rate dynamics are nonlinear, interest rates depend on lagged monetary and consumption shocks, and the price of risk is not a constant multiple of interest rate volatility. We find that habit persistence can help reproduce the nonlinearity of the spot rate process, the documented deviations from the expectations hypothesis, the persistence of the conditional volatility of interest rates, and the lead-lag relationship between interest rates and monetary aggregates.

IN THIS PAPER, we propose a new class of models for the term structure of interest rates that link the fundamentals of a monetary economy with habit formation and the dynamics of the yield curve. We begin by providing testable restrictions on how the dynamics of the nominal yield curve depend on both the habit stock and monetary factors. We then use data on nominal bonds to study whether habit persistence can help explain some of the empirical regularities highlighted by the existing term structure literature.

Both the economics and the psychology literatures stress the importance of interpersonal effects and time nonseparabilities in consumption choices. Starting with Duesenberry (1949) and Veblen (1899), who argue that consumers imitate each others' purchases to conform to the expectations of the people in their reference group, a large literature uses preferences assuming some of these features to address a variety of questions ranging from criminal behavior, business cycles, rational addiction, and asset pricing. With regards to the latter, preferences with habit persistence have been found useful to explain a number of asset pricing empirical regularities. Constantinides (1990), Stambaugh and Kandel (1991), Abel (1990), and Campbell and Cochrane (1999), for instance, argue that habit formation can help explain the large realized excess

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equity returns, and that the assumption of time separability plays an important role in the difficulty traditional asset pricing models have in reproducing the empirical regularities of equity returns. However, very little is known with regards to the implications of these models in terms of the cross-sectional and time-series properties of the term structure of interest rates and whether they can help explain some of the empirical features found in the data.

With regards to the empirical regularities, Duffee (2002) finds that across the maturity spectrum, the unconditional mean excess return to bonds is small relative to the variation in conditional mean excess returns, and the conditional volatility of yields is very persistent. Moreover, linear projections of bond yield changes on the slope of the yield curve generate large and negative Campbell-Shiller (1991) slope coefficients. This result is robust across various time periods, liquidity risk events (see Buraschi and Menini (2002)), and statistical methods. Dai and Singleton (2002) find that "key to matching the empirical findings in Fama and Bliss (1987) and Campbell and Shiller (1991) are parameterizations of the market price of risk that let the risk factors affect the market price of risk directly, and not only through their factor volatilities." Cheridito, Filipovic, and Kimmel (2005) investigate a class of flexible arbitrage-free specifications of the market price of risk.

Second, there is mounting evidence against the Fisher neutrality assumption. Benninga and Protopapadakis (1983), Fama (1990), Evans (1998), and Bodoukh (1993) find that the inflation rate is negatively related to the real interest rate in terms of both realized changes and expected values. In addition, Fama (1976, 1990), Fama and Gibbons (1982), and Marshall (1992) find that real returns on nominal bonds decline when inflation increases. Moreover, Chen, Roll, and Ross (1986) find that assets that are positively correlated with inflation earn a lower risk premium.[1]

Third, both real and nominal interest rates appear to be correlated with past (detrended) levels of output and money (Fiorito and Kollintsas (1994), Chari, Christiano, and Eichenbaum (1995), King and Watson (1996)). When the role of money is assumed away, it is hard to explain the correlation between asset returns and money growth. Marshall (1992) shows, however, that when money is introduced into the model to facilitate transactions, the negative correlation between inflation and stock returns and the positive correlation between money growth and asset returns can become equilibrium properties. Our paper follows this path and investigates an economy in which money facilitates consumption transactions.

Fourth, Conley et al. (1997) argue that "although linear specifications are convenient for deriving and estimating explicit models of the term structure of interest rates, from the viewpoint of data description it is important to specify the short-term rate drift and possibly the diffusion in more flexible ways."[2]

[1] Fama (1981) finds evidence of inflation nonneutrality also in the stock market as real stock returns are negatively correlated with inflation. Moreover, in the medium and long term, real gross domestic product is negatively affected by an increase in inflation (Fama and Gibbons (1982), Bodouk (1983), and Harvey (1988)).

[2] Conley et al. (1997), p. 526.

They estimate the stationary density of the Federal funds rate and find evidence of nonlinearity in the short-term rate using semiparametric methods. Similar results are also discussed by Ait-Sahalia (1996, 1999) using different econometric methods.

We use the above insights to develop and estimate a simple structural model in which some of these features arise in equilibrium. We explore a tractable monetary version of an exchange economy with external habit formation in which the term structure of interest rates has the following properties in equilibrium. First, the market price of risk is not a constant multiple of interest rate volatility. The term premium is state dependent so that the model can accommodate deviations from the expectations hypothesis. Second, the inflation risk premium is positive and time varying, so that the model can allow for deviations from the Fisher hypothesis. Third, the model induces a lead-lag relationship between nominal interest rates and money, and between nominal interest rates and consumption. Fourth, yield to maturities are not affine in the state variables. In particular, the spot interest rate has a nonlinear drift that captures some of the empirical properties described in Ait-Sahalia (1996, 1999), and Conley et al. (1997).

Our model builds on the work by Campbell and Cochrane (1999), who investigate an economy with nonseparable preferences and constant interest rates. In this model, the representative agent's current utility depends not only on his own current consumption, but also on the history of aggregate consumption.[3] This generates a wedge between relative risk aversion and the intertemporal marginal rate of substitution. Negative endowment shocks, pushing current consumption toward the habit stock, make investors more risk averse. Therefore, during recessions, asset prices must drop more than in a time-separable economy in order to reflect the higher state-dependent risk premium. An important advantage of the extrinsic model specification is to avoid the implied excess interest rate volatility and negative Arrow-Debreu state price density, which is sometimes present in internal habit models (see Chapman (1998) for a discussion). Menzly, Santos, and Veronesi (2004) investigate the implications of the model for the cross section of equity returns. We extend Campbell and Cochrane (1999) and Menzly et al. (2004) in two important ways. First, we allow interest rates to be stochastic. Second, we consider a monetary economy that supports positive monetary holdings in equilibrium. This has some important implications. Specifically, the Fisher relationship does not hold, thus assets that are positively correlated with inflation earn lower returns and the documented link between money growth and nominal interest rates is an equilibrium feature of the model. Moreover, since preferences are nonseparable, the model can account for a persistent correlation between money growth and interest rates. However, the monetary aspect of the economy requires that we solve for the inflation rate as an endogenous stochastic process. This is a nontrivial step that links inflation and nominal interest rates to the state variables governing the

[3] In an earlier working paper draft of the published article, Campbell and Cochrane (1998) discuss the implications of an economy with stochastic interest rates.

real side of the economy. We derive closed-form solutions for both the real and nominal term structures of interest rates, the inflation risk premium, and conditional yield volatilities. We find that the model-implied equilibrium process of interest rates is nonlinear. We then estimate the structural model using data on the U.S. nominal Treasury bonds from January 1960 to December 2000, and address the following questions.

First, to what extent does a model with habit persistence link term structure dynamics with economic fundamentals for reasonable values of the structural parameters? We find, based on asymptotic generalized method of moments (GMM) tests, that the joint moment restrictions of the model on both bond yields and macroeconomic time series such as inflation and monetary holdings are not rejected. Moreover, when we investigate the cross-sectional implications of the model, we find that the implied median absolute error for the 1-year yield to maturity is 14.6 basis points and for the 5-year bond is 13.1 basis points. The model is also quite accurate in fitting the conditional volatility of yields. We run a regression of the squares of yield changes on the model-implied conditional second moment of yield changes. We cannot reject the null hypothesis that the intercept is zero at any maturity and that the slope coefficient is one for maturities between 3 months and 3 years. The null hypothesis is rejected, however, for maturities equal to or above 5 years.

Second, does habit persistence help explain the Campbell-Shiller (1991) expectations puzzle? We compute the model-implied slope coefficients of a regression of changes in yields on the slope of the yield curve. These coefficients, known as the Campbell and Shiller (1991) coefficients, are considered important statistics describing the conditional second-moment properties of a term structure model (see Dai and Singleton (2000) and Duffee (2002)). We find that the model-implied Campbell-Shiller slope coefficients are negative and increasing (in absolute value) with the horizon. The magnitude of these coefficients matches those found in the expectations hypothesis literature. At 2- and 5-year horizons, the empirical Campbell-Shiller slope coefficients are -0.95 and -1.72, while the model-implied coefficients are -0.339 and -1.274. We find that the time variation in the habit stock plays a crucial role in explaining the time variation in the forward premium.

Third, how large is the inflation risk premium and is it time varying? Increasing empirical evidence shows that nominal interest rates are not consistent with the Fisher hypothesis, which assumes that nominal interest rates are equal to real interest rates plus expected inflation. For instance, on average the spread between the yields of nominal bonds and index-linked bonds is larger than realized inflation and its dynamics are only partially explained by changes in expected inflation. In our model the spread between nominal and real interest rates includes a state-dependent inflation risk premium. We find that the inflation risk premium accounts for about one-fourth of the spread between nominal and real interest rates. The inflation risk premium is upward sloping and time varying. The average inflation risk premium is 44 basis points for an 8-year horizon and ranges between 20 and 90 basis points.

Fourth, to what extent does the time variation in the inflation risk premium explain the rejection of the expectations hypothesis? We regress the forward premium on the inflation risk premium and find that a large component of the time variation in the forward premium is due to the time variation in the inflation risk premium. The results hold for any horizon between 6 months and 10 years.

Fifth, to what extent does the model explain the nonlinearity of the short-term interest rate? We find that the model implies a nonlinear spot interest rate process that is similar to the one estimated using both the semiparametric method of Conley et al. (1997) and the nonparametric method of Ait-Sahalia (1999). Moreover, our results show that habit persistence helps explain the hump-shaped response of consumption to monetary shocks and the lead-lag correlation between real interest rates and output (see also Fuhrer (2002) and Boldrin, Christiano, and Fisher (2001) for similar results based on calibration methods).

Related Literature. The model in this paper links two separate streams of the literature: monetary models of the term structure of interest rates and business cycle models with habit formation. Important contributions to the monetary literature include Bakshi and Chen (1996) and Marshall (1992). Marshall (1992) shows that when money is introduced in the model to facilitate consumption transactions, it is possible to reproduce the negative correlation between inflation and stock returns. Bakshi and Chen (1996) and Buraschi and Jiltsov (2005) derive implications for the term structure of interest rates endogenizing inflation in a money-in-the-utility-function model. However, these models assume time-separable preferences. We explore a setting in which this assumption is relaxed.

Sundaresan (1989), Constantinides (1990), Abel (1990), and Detemple and Zapatero (1991) are among the first to relax the time-separability assumption and study preferences with habit formation. Constantinides (1990) discusses a rational expectations model in which habit is "intrinsic," that is, rationally anticipated by the investor when making optimal consumption and investment decisions. He shows that habit persistence induces a more realistic average equity risk premium and helps to reduce the implied risk-free rate (risk-free rate puzzle).[4] In his model, habit persistence allows stock prices to be rationally volatile even if the consumption process is smooth. Menzly et al. (2004) investigate the cross-sectional expected equity returns implications of a Campbell and Cochrane (1999) economy and propose a structural explanation for the observed predictability in stock returns. Other contributions in this literature include Dunn and Singleton (1986), Ferson and Constantinides (1991), Bakshi

[4] Mehra and Prescott (1985) and Hansen and Jagannathan (1991) show that the traditional consumption-based CAPM is not consistent with the observed equity premium for reasonable levels of the risk-aversion coefficient. Additional studies show that, with respect to the arrival of new information, the time series of aggregate consumption is too smooth, the real interest rate too low, and the volatility of stock prices too high to be reconciled with models with traditional complete-markets time-separable preferences.

and Chen (1996b), Daniel and Marshall (1997), Dai (2002), Wachter (2005), Carroll, Overland, and Weil (2000), and Heien and Duhram (1991).

The empirical evidence finds evidence in support of habit persistence if consumption is allowed to be locally substitutable. Heaton (1995) finds that the stochastic discount factor (SDF) of a model with long-term habit persistence is consistent with the Hansen and Jagannathan (1991) bounds and can match the autocorrelation pattern of the monthly returns of stock and Treasury bills. However, the model finds it difficult to fit both the volatility of stock returns and the equity premium at the same time. Using aggregate consumption data, Ferson and Constantinides (1991) find evidence of habit formation in stock returns using Euler restrictions at monthly, quarterly, and annual frequencies. Some new direct studies of preferences with habit formation are based on longitudinal consumption data. Dynan (2000) tests and rejects habit formation in consumption using annual household food consumption (PSID) data. Ravina (2005) uses a data set consisting of individual-specific U.S. credit card accounts and finds evidence of habit persistence in household consumption once individual heterogeneity and credit constraints are allowed.

The paper is also related to Bekaert, Engstrom, and Grenadier (2004), who explore the role of preference shocks in a multifactor affine model in an effort to explain the empirical regularities of both equity returns and interest rates, and to Wachter (2005), who studies the link between short-term real rates and lagged average real consumption growth rates and shows that habit persistence helps explain this relation. An important difference in this paper is that, while Wachter (2005) focuses on a real economy, this paper provides closed-form solutions for the nominal term structure in a (continuous-time) monetary economy. This result is useful as most of the empirical evidence and available data are on nominal bonds.

Our paper proceeds as follows: Section I presents the model. Sections II and III discuss its asset pricing implications. Section IV describes the data set. Section V presents the econometric methodology and Section VI summarizes the empirical results. Section VII discusses the nonlinear properties of the model-implied spot interest rate. Section VIII tests the model's implications for the lead-lag relationships among interest rates, consumption, and money. Section IX shows the extent to which the model can explain the Campbell-Shiller (1991) expectations puzzle, and Section X investigates the properties of the model-implied inflation risk premium and compares it to the empirical evidence. Section XI concludes. All proofs are in the Appendix.

I. The Model

We study a representative agent endowment economy with habit formation. Real monetary holdings are assumed to provide a transaction service by reducing the total amount of resources X_t needed to achieve a given level of net consumption C_t. Thus, in this economy, money is held because of its positive marginal productivity. Clearly, one may desire to work with a simpler real model, abstracting from the nominal side of the economy. However, a real model would not give realistic implications for nominal interest rates. Since most of the available empirical evidence is on nominal Treasury bonds, this would be a shortcoming. Moreover, a monetary framework is necessary to take into account the structural interaction between the monetary and real growth rates of the economy and the different dynamics of the real and nominal yield curves.

ASSUMPTION 1:

(a) *(Preferences). The representative agent is affected by external habit formation, H_t. The agent chooses his consumption plan and nominal monetary holdings to maximize his expected value of utility, $u(X, H)$,*

$$E_0 \int_0^{\infty} e^{-\rho t} \log(X_t - H_t)\, dt, \;\; \rho > 0. \tag{A1}$$

(b) *(Transaction Costs). The consumption of X_t entails a proportional transaction cost $1 - \psi$. Given a level of gross consumption C_t, the level of net consumption is $X_t = \psi C_t$, with $\psi(C_t, \frac{M_t}{P_t}) = \psi_0(\frac{M_t/P_t}{C_t})^{\gamma}$, $0 \le \gamma \le 1$, and ψ_0 being a normalization factor such that $0 \le \psi \le 1$. M_t and P_t are the nominal amount of monetary holdings and the general price level, respectively.*

This assumption implies that money reduces the transaction costs of obtaining the desired level of consumption. Thus, although money does not yield any interest, money is demanded in equilibrium.[5] A similar approach is used to investigate monetary equilibria in different settings by Marshall (1992) and Bekaert (1996). In the case of time-separable preferences, Feenstra (1986) shows that money-in-the-utility models are equivalent to assuming that money facilitates consumption transactions. This specification is also related to Bakshi and Chen (1996a), who consider an economy with a money-in-the-utility function.[6]

The stochastic sequence of habits H_t is regarded as exogenous by each agent and specified implicitly as a function of the past aggregate consumption and monetary holdings. The existence of transaction costs coupled with habit persistence makes monetary shocks have a persistent effect on future asset prices and interest rates. The intuition is simple. With monetary transaction costs, current money shocks affect ψ and therefore current marginal utility and optimal current consumption. However, because current consumption affects the habit stock H_t, current money shocks affect future marginal utilities and therefore future asset prices. This link is very important in order to reproduce the persistence found in the data.

The equilibrium level of risk aversion is state dependent: Individuals become more risk averse in bad times (when consumption is low relative to its past

[5] In the period 1905 to 2004, money velocity $\frac{C_t}{M_t/P_t}$ ranged between one and two. Thus, a value of $0 < \psi_0 < 2^{\gamma}$ guarantees that the *realized* value of the transaction cost function is $0 < \psi < 1$.

[6] The previous specification can also be thought of as one in which the agent consumes a composite good $X = C^{1-\gamma}\, C_*^{\gamma}$, with C_* being a cash good subject to a cash-in-advance constraint $C_* = M/P$.

values) than in good times (when consumption is high relative to its historical levels). Thus, habit H_t affects the way in which consumption shocks change the level of risk aversion in a state-dependent fashion. This property allows one to investigate a (potentially) more flexible term structure model, even when consumption growth is independently and identically distributed (i.i.d.), without giving up tractability. We follow Campbell and Cochrane (1999) and Menzly et al. (2004) and model the habit H_t as an external process in terms of the process $S_t = \frac{X_t - H_t}{X_t}$, the surplus-consumption ratio, or equivalently, its inverse Y_t. Clearly, given an equilibrium process for X_t, there is a one-to-one mapping between Y_t and H_t, so that the choice of which one to model is immaterial. However, since the local curvature of the utility function $-X_t \frac{u_{xx}}{u_x}$ is equal to $\frac{1}{S_t}$, it is more convenient to model Y_t directly.

ASSUMPTION 2 (Habit Formation): *Let the net surplus-consumption ratio be $S_t = \frac{X_t - H_t}{X_t}$ and let $Y_t = \frac{1}{S_t}$. Y_t follows a stochastic mean reverting process*

$$dY_t = k_y(\theta_y - Y_t)\,dt - (Y_t - \lambda)\sigma_y\,dW_t \tag{A2}$$

initialized at $Y_0 > \lambda > 1$.

The Y_t process is bounded below by λ. Thus, when $\lambda \geq 1$ specification (*A2*) ensures that the habit H_t is always positive. This condition also implies that $X_t > H_t$ so that the marginal utility is always finite and positive. Further, Y_t is mean reverting to θ_y and its stochastic innovations are driven by *unexpected* innovations in net consumption X_t. Thus, the dynamics of Y_t are a function of both the gross consumption endowment shocks and the liquidity shocks affecting their service flow. The stochastic process in (*A2*) is also used by Menzly et al. (2004) to study the cross-section of (real) expected equity returns. This diffusion process is not affine since the local variance is quadratic in the state variable. It is easy to verify, however, that the drift and diffusion coefficients satisfy global Lipschitz and growth conditions, which imply the existence and uniqueness of a strong solution to (*A2*).[7] Of course, for the process to be stationary additional restrictions are required. The stationarity of the process depends on the boundary behavior of Y_t. The following proposition discusses the conditions under which the process (*A2*) admits a unique strong solution and a stationary density p, such that when the diffusion is initialized with a density p, the diffusion is stationary with stationary density p (see Hansen and Scheinkman (1995), p. 774 and Karlin and Taylor (1981), p. 221).[8] The proposition also provides the conditions under which the process is square-integrable.

[7] For the statement of Ito's classical results of strong form existence and uniqueness using global conditions, see Oksendal (2002), theorem 5.5); for a statement using local Lipschitz, see Karatzas and Shreve (1991), theorem 2.5, p. 287; for even weaker conditions see the Yamada–Watanabe theorem (Karatzas and Shreve (1991), theorem 2.13 p. 291).

[8] See Ait-Sahalia (1996), appendix 1; see also Conley et al. (1997), assumptions 1 and 2, p. 532 and 533.

PROPOSITION 1:

A. *(Stationarity). If the process (A2) satisfies condition (C1),*

$$\theta_y > \lambda > 1, \quad k_y > 0, \quad \text{and } Y_0 > \lambda, \tag{C1}$$

then Y_t does not explode and its unique strong solution is

$$Y(t) = \lambda + \omega_t^{-1}\left[(Y_0 - \lambda) + k(\theta - \lambda)\int_0^t \omega_s\, ds\right]$$

$$\omega_t = \exp\left\{\left(k + \frac{1}{2}\sigma_y^2\right)t + \sigma_y(W_t - W_0)\right\}.$$

Moreover, the boundaries of $(\lambda, +\infty)$ are both entrance boundaries and the stationary density p(y) of Y is Inverse-Gamma

$$p(y) = \mathcal{N}\cdot(y-\lambda)^a \exp\left(\frac{b}{y-\lambda}\right),$$

with $\mathcal{N} = \frac{(-b)^{-1-a}}{\Gamma(-1-a)}$, $a = -2(1 + \frac{k_y}{\sigma_y^2})$, and $b = -2\frac{k_y(\theta_y - \lambda)}{\sigma_y^2}$.

B. *(Square-integrability). If, in addition to (C1), the following condition is also satisfied, then $E|Y|^2 < \infty$, and the process is square-integrable*[9]

$$2k_y - \sigma_y^2 > 0. \tag{C2}$$

Part A of Proposition 1 shows that the inverse of the surplus-consumption ratio $Y(t)$ is a weighted average of the lagged shocks W_s, for $0 \le s \le t$. The persistence of the shocks depends on $(2k + \sigma_y^2)$; when this is positive, the solution converges. Moreover, Part A shows that when the process is properly initialized, mean reversion (i.e., $k > 0$) is a sufficient condition for the two boundaries to be entrance boundaries.[10] Part B shows that in order for the process to have bounded unconditional second moments, mean reversion is not sufficient and a stricter condition is required. A full characterization of the conditional moments is given in Proposition 3.

The parameter λ is the lower bound of Y_t, so that $(0, \frac{1}{\lambda})$ is the support of the surplus ratio S_t. Campbell and Cochrane (1999) restrict λ to yield a constant real interest rate. Since our focus is on modeling the dynamics of the term structure of interest rates, we do not impose such a restriction. Instead, we require $\lambda \ge 1$ to avoid negative habit formation.

The growth rate of the aggregate consumption endowment process is i.i.d. and follows the process

$$\frac{dC_t}{C_t} = \mu_c\, dt + \sigma_c\, dW_t^c.$$

[9] In this case the process is also said to be stationary in a wide sense. For a definition of strict and wide sense stationarity see Liptser and Shiryaev (2001), p. 23.

[10] For the sake of comparison, it is insightful to observe that in the case of square-root diffusions, Feller (1951) shows that zero is an entrance boundary for the process if $2k\theta > \sigma^2$.

The assumption of an i.i.d. consumption growth rate process is motivated by a large body of empirical evidence that argues that deviations from this assumption, which is consistent with the permanent income hypothesis (PIH), are small. At the aggregate level, the autocorrelation of the U.S. consumption[11] at the quarterly frequency is 0.22 over the 1947 to 1996 period and it becomes -0.117 over the 1891 to 1995 period.[12] This assumption is also consistent with the empirical literature on the PIH based on household data.[13] Thus, following Campbell and Cochrane (1999), we exogenously constrain the consumption growth process to be i.i.d. and investigate whether it is possible to generate the required autocorrelation in the SDF via a parsimonious habit specification.

Stock and Watson (1989) find that the M1 money supply process in the United States can be described as a stationary process around a positive deterministic time trend.[14] Thus, we consider a process for the money supply that is given by two components, a deterministic (exponential) trend, μ_M, and a stochastic deviation from this trend, L_t, which can be thought of as the detrended inverse of the money supply, that is,

$$d \ln M_t = \mu_M dt - d \ln L_t, \quad \mu_M > 0. \tag{1}$$

ASSUMPTION 3 (Money Supply): *Let L_t be the aggregate liquidity shock generated by n factors ℓ_{it} following the stochastic process*

$$d\ell_{it} = k_{\ell_i}(\theta_{\ell_i} - \ell_{it})\,dt + \sigma_{\ell_i}\ell_{it}\,dW_t^{\ell_i}, \quad \text{with } E\left(dW_t^{\ell_i}\,dW_t^c\right) = \rho_{\ell_i,c}\,dt \tag{A3}$$

and

$$L_t = \sum_{i=1}^{n} \ell_{it}, \quad \text{with } E\left(dW_t^{\ell_i}\,dW_t^{\ell_j}\right) = 0.$$

This particular specification of the ℓ_{it} process is of special interest for two reasons. First, Nelson (1990) shows that this process is the continuous-time limit of the GARCH(1,1)-M studied by Engle and Bollerslev (1986), which has been extensively used to model empirically log-excess equity returns but never to investigate the implications of a term structure model. Second, it will be shown that the interest rate dynamics depend on the ℓ_t process and that in

[11] For the U.K. the autocorrelation is -0.017 (1970 to 1996); for Canada it is 0.113 (1970 to 1996).

[12] See "Asset Prices, Consumption, and the Business Cycle," in *Handbook of Macroeconomics*, edited by John B. Taylor and Michael Woodford.

[13] Attanasio and Weber (1995) use a time series of cross sections from the Consumer Expenditure Survey (1980–1990) to test the PIH after controlling for several variables that are likely to affect family composition and labor supply behavior over the business cycle. They use a robust instrumental variables technique to control for the potential endogeneity of the explanatory variables. They find that the excess sensitivity to labor income disappears after controlling for these effects (see Table 3) and conclude that consumption growth displays very modest autocorrelation.

[14] Stock and Watson (1989) also investigate a specification for the log of the money supply with a quadratic time trend. To simplify the derivation, we consider the simpler case with a linear time trend.

equilibrium the quadratic local variance of ℓ_t produces a nonlinear drift in interest rates. We investigate the extent to which this particular model-implied nonlinearity in interest rates is close to that found in the data by Conley et al. (1997) and Ait-Sahalia (1999).

Since ℓ_t is a shifted version of Y_t, with a boundary set at zero, the conditions for the existence of a stationary density and for square-integrability are analogous to Conditions (*C1*) and (*C2*) with $\lambda = 0$. Thus, the process ℓ_t is stationary if $\theta_\ell > 0$, $k_\ell > 0$, and $\ell_0 > 0$ and square-integrable if $2k_\ell - \sigma_\ell^2 > 0$.[15] The closed-form solution of the stationary density can be obtained from the result of Proposition 1a setting $\lambda = 0$.

Using Ito's rule, it follows that in a one monetary factor economy, $\frac{dM_t}{M_t} = ((\mu_M + k_\ell + \sigma_\ell^2) - \frac{k_\ell \theta_\ell}{\ell_t})dt - \sigma_\ell \, dW_t^\ell$. The expected growth rate of the nominal money supply can be either positive or negative depending on the level of the state variable ℓ_t. The drift of the money growth rate is positive (negative) if ℓ_t is above (below) $\frac{k_\ell \theta_\ell}{\mu_M + \sigma_\ell^2 + k_\ell}$. However, the nominal stock of money supply is always positive. For expositional simplicity and to streamline the notation, whenever clear from the context we drop the subscript i from ℓ_t.

Since the innovations of the inverse-surplus ratio $\sigma_y \, dW_t$ must be functions of the same Brownian innovations driving the state variables of the economy, namely, $dW_t = [dW_t^c, \, dW_t^{\ell_i}]$, we simply define $\sigma_y dW_t \equiv [\sigma_{cy} dW_t^c - \sigma_{\ell y}' dW_t^\ell]$, with $\sigma_y^2 \equiv \sigma_{cy}^2 + \sum_i (\sigma_{\ell_i y}^2 - 2\rho_{\ell_i, c} \sigma_{cy} \sigma_{\ell_i y})$. When $\sigma_{cy} > 0$, a negative consumption shock increases the inverse surplus-consumption ratio, thus inducing an increase in the investor's implied (local) risk aversion. The second component comes from liquidity shocks. Due to habit formation in consumption, the transaction service generated by liquidity shocks at time t affects the marginal utility of consumption in future time periods as well. If monetary holdings do not provide a transaction service, the dynamics of the habit stock are driven only by the past real consumption. We let the data indicate the empirical magnitude of $\sigma_{\ell y}$.[16]

In addition to conditions (*C1*), (*C2*) and $\rho > 0$, we require additional parameter restrictions to ensure $\mathrm{cov}(dH_t, dC_t)/dt > 0$, so that positive innovations in consumption increase the habit. Let $c^* = \max_i \sigma_{\ell_i} \sigma_c \rho_{c, \ell_i}$. It is easy to show that a sufficient condition is that $[(1 - S_t)\sigma_c^2 - (1 - S_t)\gamma c^* - S_t \sigma_y (1 - \lambda S_t)\psi_c] \geq 0$, with $\psi_c = \sigma_{yc}\sigma_c - \sigma_{y\ell_i}\sigma_c \rho_{\ell_i, c}$. The lower bound of this condition is reached at $S_* = v\frac{1}{\sigma_y}$, with $v = \frac{\sigma_y \psi_c + \sigma_c^2 - \gamma c^*}{2\lambda\psi_c}$. Substituting S_* back, we find that $\mathrm{cov}(dH_t, \, dC_t)/dt > 0$ if $\sigma_y \geq \frac{v\sigma_c^2 - v\gamma c^* + \lambda \psi_c v^2}{\sigma_c^2 - \gamma c^* - v\psi_c}$.[17]

[15] In the case of a Cox, Ingersoll, and Ross (1985b) square-root processes, the equivalent condition for zero being nonattracting is $2k\theta - \sigma^2 > 0$.

[16] In the monetary literature, it is common to interpret the long-term mean of the monetary policy as the target of a Taylor rule in which the monetary authority adjusts the money supply according to the deviations of some observable economic aggregates from their target levels. We refer to Buraschi and Jiltsov (2005) for a discussion of a monetary model of the term structure with an explicit two-factor Taylor rule.

[17] The condition requires $a\alpha^2 + b\sigma + c \geq 0$, with $\hat{a} = \frac{1}{4}\frac{\sigma_L \psi_L}{\lambda}$, $\hat{b} = (\frac{1}{2\lambda} - 1)\sigma_c \sigma_L \rho_{c,L}$, $\hat{c} = \sigma_L q_1 (\sigma_c \rho_{c,L} - \lambda \psi_L)$, and $q_1 = \frac{\sigma_c \rho_{c,L}}{2\lambda \psi_L}$, where $\psi_L = \sigma_{yc}\sigma_L \rho_{c,L} - \sigma_{yL}\sigma_L$.

A. Aggregation in Heterogeneous-Agent Economies

Since the aggregation properties of economies with this class of habit preferences are not completely known, in this section we briefly investigate the link between the competitive equilibrium prevailing in a heterogeneous-agent economy and the representative agent equilibrium. We consider two heterogenous economies. In the first economy, all agents have identical logarithmic preferences (as in our model) but different endowments. In the second economy, agents have heterogeneous preferences.[18]

Case 1. (Heterogeneous endowment)—When markets are dynamically complete it is possible to show that equilibrium asset prices in a heterogeneous-agent economy are the same as those prevailing in a representative agent economy in which the single agent's consumption is equal to the total available resources $X_t = \sum_i x_t^i$ and $H_t = \sum_i h_t^i = X_t - \sum_i s_t^i x_t^i$. When markets are dynamically complete, the SDF $\xi_{t,T}^i = e^{-\delta(T-t)} \frac{u'(x_T^i - h_T^i)}{u'(x_t^i - h_t^i)}$ of each agent in the heterogeneous economy is unique and identical across agents, so that $\xi_{t,T}^i = \bar{\xi}_{t,T}\ \forall i$. With logarithmic preferences, this implies that $[e_{t,T}^{\delta(T-t)} \bar{\xi}_{t,T}](x_T^i - h_T^i) = (x_t^i - h_t^i)$. Since the term in square brackets is agent independent, we can aggregate across agents. We then obtain that $\bar{\xi}_{t,T}$ is equal to the intertemporal marginal rate of substitution of an agent with logarithmic preferences who consumes the aggregate endowment, that is, $u(X_t, H_t) = \log(X_t - H_t)$. Since $s_t^i x_t^i = x_t^i - h_t^i$ and in equilibrium $x_t^i - h_t^i = x_t^j - h_t^j$, then $\sum_i s_t^i x_t^i = X_t - H_t$ and thus the representative agent's surplus-consumption ratio is a weighted average of the individual surplus ratios, that is $S_t = \sum_i \omega_i s_t^i$, where the weights ω_i are equal to the individual consumption shares $\omega_i = (x_t^i / X_t)$.

Case 2. (Heterogeneous preferences)—Since for analytical convenience in the rest of the paper we assume the existence of a representative agent with logarithmic preferences, in what follows we study whether a disaggregated economy with multiple agents and heterogeneous constant relative risk aversion (CRRA) preferences admits a representative agent aggregation. The case of a representative agent with logarithmic preferences is a special case. Let us assume that each agent has preferences of the type $u_t = e^{-\delta t} \frac{1}{1-\gamma}(x_t(\gamma) - h_t(\gamma))^{1-\gamma}$ and $u_t = e^{-\delta t} \ln(x_t - h_t)$ for the agent with $\gamma = 1$.

In a heterogeneous economy, the social planner allocates the aggregate endowment X_t across agents to achieve Pareto efficiency. Let $\omega(\gamma)$ be the social weight attributed to the agent of type γ. Since there is no intertemporal transfer of resources via a production technology, at each time t the social planner maximizes the objective function

[18] An alternative nonpsychological interpretation of models with habit persistence is discussed by Chetty and Szeidl (2003) in the context of economies with forced consumption commitments.

$$\sup_{x_t(X_t,h_t;\gamma)} \int \omega(\gamma)\frac{1}{1-\gamma}(x_t(\gamma)-h_t(\gamma))^{1-\gamma}d\gamma \quad \forall t$$

subject to the resource constraint $\int x_t(\gamma)\,d\gamma \leq X_t$. Let z_t be the Lagrange multiplier associated with the resource constraint. The first-order conditions are then $\omega(\gamma)(x_t(\gamma)-h_t)^{-\gamma} = z_t$ and $\int x_t(\gamma)\,d\gamma = X_t$. Solving for x_t and substituting in the resource constraint we have $X_t - H_t = \int \omega(\gamma)^{1/\gamma} z_t^{-1/\gamma}\,d\gamma$. The Lagrange multiplier z_t is an implicit function of the aggregate surplus consumption $X_t - H_t$. Let $z_t = \varphi(X_t - H_t)$ be the solution of the previous implicit function. Asset prices depend on the representative agent's SDF $\xi_{t,T}$, which is equal to $e^{-\delta(T-t)}\,(z_T/z_t)$. Substituting the solution for the Lagrange multiplier, we have $\xi_{t,T} = e^{-\delta(T-t)}\frac{\varphi(X_T-H_T)}{\varphi(X_t-H_t)}$.

Two things should be noticed. First, the representative agent's SDF is a deterministic function of excess aggregate consumption, $X_T - H_T$. The representative agent inherits the external habit preferences of the heterogenous agents as the argument of φ is the aggregate surplus consumption. The function φ, however, is not an arithmetic average of each agent's power functions.

Second, from the first-order conditions it is possible to obtain that in equilibrium the consumption share of an agent of type γ is equal to $\frac{x_t(\gamma)}{X_t} = \frac{h_t(\gamma)}{X_t} + \omega(\gamma)^{1/\gamma} y_t(\gamma)$, where $y_t(\gamma) = \frac{z_t^{-1/\gamma}}{X_t}$ is the solution to $S_t = \int \omega(\gamma) y_t(\gamma)\,d\gamma$. Since, under condition (*C1*), S_t is stationary, then each agent's consumption share $\frac{x_t(\gamma)}{X_t}$ is stationary. No single agent dominates the economy in the long run. This contrasts with heterogenous-agent economies with standard CRRA agents in which the economy eventually becomes dominated by the least risk-averse agent (Wang (1996)). The reason for the different asymptotic behavior is due to the effect of the habit on the cross section of marginal utilities. In a traditional economy, as the economy grows, the agent with lower risk aversion experiences higher marginal utility of consumption. He is therefore allocated a progressively higher share of the total endowment. This does not occur in the habit economy since consumption growth also has the effect of increasing the habit stock, which reduces the marginal utility of consumption, and this reduction is larger the lower the risk-aversion coefficient. Thus, habit persistence is an interesting and effective way to achieve stationarity in the distribution of consumption allocation in the disaggregated heterogeneous-agent economy.

In order to obtain closed-form solutions, our analysis below focuses on economies that support $\varphi(X_T - H_T) = (X_T - H_T)^{-1}$, that is, economies supporting a logarithmic representative agent.

B. The Equilibrium Price Level

We begin by solving for the general price level in equilibrium. We then use the result to solve for equilibrium asset prices. Let us define the real stock of money as $m = M/P$. Notice that in equilibrium the following transversality condition needs to hold in order for an interior condition to exist.

Transversality Condition

$$\lim_{T\to\infty} e^{-\rho T} E_t \left[u_x(X_T, H_T) \frac{\partial X_T}{\partial C_T} \frac{1}{P_T} \right] = 0. \tag{2}$$

If (2) were not satisfied, a small reduction in consumption at time t would yield a large discounted marginal utility by using money as a store of value and delaying consumption up to time T.

To solve for the equilibrium price level, we follow Bakshi and Chen (1996) and consider the marginal decision between consumption and monetary holdings. Since the inverse of the general price level $\frac{1}{P_s}$ is the relative price of the monetary holdings with respect to consumption, a marginal reduction in one unit of real monetary stock yields a marginal utility reduction of $u_x(X_t, H_t)\frac{\partial X_t}{\partial C_t}\frac{1}{P_t}$. The marginal revenue, over a dt interval of time with $s = t + dt$, is $E_t e^{-\rho dt} u_x(X_s, H_s)(\frac{\partial X_s}{\partial m_s} dt + \frac{\partial X_s}{\partial C_s})\frac{1}{P_s}$. The first term is the service flow of the monetary stock; the second term comes from the increase in consumption in the later period. Since in equilibrium the agent must be indifferent between reducing consumption or the money stock, the following Euler restriction must be satisfied $\forall\, dt$:

$$u_x(X_t, H_t)\frac{\partial X_t}{\partial C_t}\frac{1}{P_t} = E_t e^{-\rho dt}\left[u_x(X_s, H_s)\left(\frac{\partial X_s}{\partial m_s} dt + \frac{\partial X_s}{\partial C_s}\right)\frac{1}{P_s}\right]. \tag{3}$$

Equation (3) can be solved forward. Using the transversality condition and taking the continuous-time limit, we obtain

$$\frac{1}{P_t} = E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{u_x(X_s, H_s)\dfrac{\partial X_s}{\partial m_s}}{u_x(X_t, H_t)\dfrac{\partial X_t}{\partial C_t}} \frac{1}{P_s}\, ds \right]. \tag{4}$$

Money is a durable stock, and thus its value reflects its service flow *in all future periods*.[19] Taking derivatives, we obtain $u_x(X_t, H_t)\frac{\partial X_s}{\partial C_s} = (1-\gamma)\frac{1}{S_t C_t}$ and $u_x(X_s, H_s)\frac{\partial X_s}{\partial m_s} = \gamma \frac{1}{S_s m_s}$. Hence, we have

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma} C_t S_t \int_t^\infty e^{-\rho(s-t)} E_t \left[\frac{P_s}{S_s M_s} \frac{1}{P_s} \right] ds \tag{5}$$

$$= \frac{\gamma}{1-\gamma} C_t S_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} \frac{1}{L_t M_t} E_t[Y_s L_s]\, ds. \tag{6}$$

Since L is the inverse of the money supply, for simplicity one could initialize $L_0 M_0 = 1$, so that $\frac{1}{L_t M_t} = e^{-\mu_M t}$. To solve for the equilibrium price level, which allows one to solve for nominal interest rates, we need to solve for the conditional

[19] Bakshi and Chen (1996) use this Euler condition to solve for a stochastic monetary equilibrium in an economy with time-separable preferences.

expected value of the product of two GARCH-Ito processes. The following lemma provides the general result for this class of processes.

LEMMA 1: *Consider a linear system of two mean-reverting GARCH-Ito processes ξ_{1t} and ξ_{2t},*

$$d\xi_{1t} = k_{\xi_1}(\theta_{\xi_1} - \xi_{1t})\,dt + (\xi_{1t} - \lambda_{\xi_1})\big[\upsilon dW_t^2 + \sigma_{\xi_1}\,dW_t^1\big]$$

$$d\xi_{2t} = k_{\xi_2}(\theta_{\xi_2} - \xi_{2t})\,dt + (\xi_{2t} - \lambda_{\xi_2})\sigma_{\xi_2}\,dW_t^2,$$

where $E(dW_t^1 \cdot dW_t^2)\,\rho dt$. The conditional expectation of their product, $q_t = \xi_{1t}\xi_{2t}$, is nonlinear in the state variables and equal to

$$E_t[q_{t+\tau}] = A_0(\tau) + A_1(\tau)\xi_{1t} + A_2(\tau)\xi_{2t} + A_3(\tau)q_t, \tag{7}$$

where the $A_i(\tau)$ are deterministic functions of the expectation horizon and are fully characterized in the Appendix.

Clearly, this result is important for asset pricing applications since the price of any contingent claim is the conditional expected value of the product of the SDF and the future cash flows. Moreover, it can be used in econometric applications to calculate conditional covariances of pairs of GARCH-Ito processes. To solve for the equilibrium price level, let $\Theta_Y = [k_y, \theta_y, \sigma_y, \lambda]$ and $\Theta_{\ell_i} = [k_{\ell_i}, \theta_{\ell_i}, \sigma_{\ell_i}, 0]$ be the vectors of structural parameters for Y and ℓ_i. We can then use Lemma 1 to solve for $E_t(Y_s\ell_{i,s})$. The equilibrium price level is

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma}\frac{C_t}{Y_t}\frac{1}{L_t M_t}\int_t^\infty e^{-(\rho+\mu_M)(s-t)}$$
$$\times\left[\sum_{i=1}^{2}\begin{pmatrix} A_{0i}(s-t;\Theta_Y,\Theta_{\ell_i}) + A_{\ell_i}(s-t;\Theta_Y,\Theta_{\ell_i})\ell_{it} + \\ A_Y(s-t;\Theta_Y,\Theta_{\ell_i})Y_t + A_{\ell_i Y}(s-t;\Theta_Y,\Theta_{\ell_i})\ell_{it}Y_t \end{pmatrix}\right] ds.$$

If conditions (*C1*) and (*C2*) are satisfied and the parameters $A_i(s)$ are bounded, the integral converges and the economy admits a monetary equilibrium with a finite market-clearing price level. Sufficient conditions are that $-k_y - k_{\ell_i} - (\sigma_{yc}\sigma_{\ell_i}\rho + \sigma_{y\ell_i}\sigma_{\ell_i}) < 0, \forall i$. We summarize the result as follows:

PROPOSITION 2: *If, in addition to conditions (C1) and (C2), $-k_y - k_\ell - (\sigma_{yc}\sigma_\ell\rho + \sigma_{y\ell}\sigma_\ell) < 0$, then the economy supports a monetary equilibrium whose price level is*

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma}\frac{C_t}{Y_t}\frac{1}{L_t M_t}\psi_t,$$

where

$$\psi_t = \sum_{i=1}^{2}\Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i})Y_t + \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i})\ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i})Y_t\ell_{it},$$

and $\Gamma(\cdot)$ *are deterministic functions of the structural parameters whose functional form is fully described in the Appendix.*

From the solution of the equilibrium price level we can determine the money velocity. This is defined as $v_t = \frac{P_t C_t}{M_t}$, thus $v_t = \frac{(1-\gamma)L_t Y_t}{\gamma \psi_t}$. Under conditions (*C1*) and (*C2*), both ℓ_{it} and Y_t are strictly positive stationary processes, and thus both the numerator and the denominator are stationary. It is therefore possible to show that a sufficient condition for v_t to be stationary is that the denominator does not cross zero inside the support of Y_t and ℓ_t. That is, a sufficient condition is that the coefficients Γ_0, Γ_1, Γ_2, and Γ_3 are nonnegative.

II. The Term Structure of Interest Rates

A. Nominal Spot Interest Rate

The nominal spot rate can be obtained from the first-order conditions observing that in equilibrium the agent must be indifferent between reducing current consumption to invest in a (nominal) riskless opportunity paying off $1 + R_t dt$ and reducing current consumption to increase the money stock. In the first case, the expected discounted increase in marginal utility is $e^{-\rho dt} E_t[u_x(X_s, H_s)\ \frac{\partial X_s}{\partial C_s}(1 + R_t dt)\ \frac{1}{P_s}]$. In the second case, the increase is $e^{-\rho dt} E_t[u_x(X_s, H_s)\ (\frac{\partial X_s}{\partial m_s} dt + \frac{\partial X_s}{\partial C_s})\frac{1}{P_s}]$. In equilibrium, these two quantities must be equal. Simplifying and taking the continuous-time limit, we obtain $R_t = \frac{u_x(X_t, H_t)\partial X_t/\partial m_t}{u_x(X_t, H_t)\partial X_t/\partial C_t}$, which yields the following relationship:

$$R_t = \frac{\gamma}{1-\gamma}\frac{C_t P_t}{M_t}.$$

Substituting the equilibrium price process in Proposition 2, we find the following relationship between the inverse surplus-consumption ratio and the instantaneous nominal interest rate:

$$R_t = \frac{Y_t L_t}{\left[\sum_{i=1}^{2}(\Gamma_{0i}(\Theta_Y, \Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y, \Theta_{\ell_i})Y_t + \Gamma_{\ell_i}(\Theta_Y, \Theta_{\ell_i})\ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y, \Theta_{\ell_i})Y_t \ell_{it})\right]}. \tag{8}$$

The nominal interest rate is nonlinear in the state variables. It depends on both the habit stock and the factors affecting the monetary aggregate. A higher expected growth in monetary holdings increases the interest rate. The higher the surplus-consumption ratio S_t (i.e., the higher the current level of consumption with respect to the habit), the higher the incentive to save for future consumption and the lower the nominal interest rate. Thus, the model implies that the lagged real money-adjusted consumption predicts both real and nominal interest rates. This empirical implication of the model is consistent with the finding of a lagged effect between changes in monetary holdings and interest rates, which is the focus of substantial interest in the monetary literature.

B. Bond Prices

Given the equilibrium price process, we can solve for the term structure of nominal and real bond prices. Let $N(t,\tau)$ and $B(t,\tau)$ be time-t prices of two pure discount bonds paying one unit of currency and one unit of the consumption good, respectively, at time $t+\tau$. The price of the second bond is equal to the price of an index-linked bond. The price of the nominal bond must satisfy the following Euler condition:

$$N(t,\tau) = e^{-\rho\tau} E_t \left[\frac{u_x(X_{t+\tau}, H_{t+\tau}) \dfrac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \dfrac{\partial X_t}{\partial C_t}} \frac{P_t}{P_{t+\tau}} \right].$$

The marginal cost of reducing consumption and purchasing a nominal bond is $N(t,\tau) u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t} \frac{1}{P_t}$. In equilibrium, it must be equal to the marginal utility of the future consumption that can be obtained from the bond investment $u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}} \frac{1}{P_{t+\tau}}$. The Euler restriction for the real bond is similar. However, since the payoff is indexed to the price level,

$$B(t,\tau) = e^{-\rho\tau} E_t \left[\frac{u_x(X_{t+\tau}, H_{t+\tau}) \dfrac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \dfrac{\partial X_t}{\partial C_t}} \right].$$

The price of the two bonds is affected by the habit via the surplus-consumption ratio via $\frac{u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t}} = \frac{C_t S_t}{C_{t+\tau} S_{t+\tau}}$. Using the result of Proposition 2 for the price level, we obtain $C_{t+\tau} S_{t+\tau} P_{t+\tau} = \frac{\gamma}{1-\gamma} e^{\mu_M \tau} L_t M_t / \Psi_{t+\tau}$, with $\Psi_{t+\tau} = \sum_{i=1}^{2} (\Gamma_{Y,i}(\Theta_Y, \Theta_{\ell_i}) Y_{t+\tau} + \Gamma_{\ell_i}(\Theta_Y, \Theta_{\ell_i}) \ell_{it+\tau} + \Gamma_{\ell_i Y}(\Theta_Y, \Theta_{\ell_i}) \ell_{it+\tau} Y_{t+\tau} + \Gamma_{0i}(\Theta_Y, \Theta_{\ell_i}))$. Hence, applying Lemma 1, we have the following result.

THEOREM 1 *(The Nominal Term Structure): The price of a nominal bond $N(t,\tau)$ at time t with time to maturity τ is given by*

$$N(t,\tau) = e^{-(\rho-\mu_M)\tau} \frac{\sum_{i=1}^{2} (\Lambda_{0i}(\tau;\Theta_Y,\Theta_{\ell_i}) + \Lambda_{Y,i}(\tau;\Theta_Y,\Theta_{\ell_i}) Y_t + \Lambda_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) \ell_{it} + \Lambda_{\ell_i Y}(\tau;\Theta_Y,\Theta_{\ell_i}) Y_t \ell_{it})}{\sum_{i=1}^{2} (\Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i}) Y_t + \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i}) \ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) Y_t \ell_{it})},$$

where

$$\Lambda_{Y,i}(\tau;\Theta_Y,\Theta_{\ell_i}) = \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i}) e^{-k_y\tau} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}),$$

$$\Lambda_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) = \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i}) e^{-k_{\ell i}\tau} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_Y(\tau;\Theta_Y,\Theta_{\ell_i}),$$

$$\Lambda_{\ell_i Y}(\tau;\Theta_Y,\Theta_{\ell_i}) = \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}),$$

and

$$\Lambda_{0i}(\tau;\Theta_Y,\Theta_{\ell_i}) = \Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_Y(\Theta_Y,\Theta_{\ell_i})\theta_y(1-e^{-k_y\tau})$$
$$+\Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i})\theta_{\ell i}(1-e^{-k_{\ell i}\tau}) + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i})A_{0i}(\tau;\Theta_Y,\Theta_{\ell_i}),$$

and both $\Gamma_{\cdot}(\cdot)$ *and* $A_{\cdot i}(\tau;\Theta_Y,\Theta_{\ell_i})$ *are deterministic functions of the structural parameters* Θ_Y *and* Θ_{ℓ_i}*. Their functional forms are given in Proposition 2 and Lemma 1, respectively.*

C. The Price of Risk

To gain further intuition with respect to the properties of the term structure of interest rates, it is instructive to study the properties of the price of risk that are implied by the model. In the term structure literature, the issue of flexible specifications of the price of risk has received considerable attention. Let ξ_t be the SDF, $\xi_t = e^{-\rho t}\, u'\,(c_t^*, M_t^*)$, so that the discounted value of any tradable asset is a martingale, $\xi_t B_t^\tau = E_t(\xi_s B_s^\tau)$. The diffusion process of the SDF must be of the form

$$\frac{d\xi_t}{\xi_t} = -r_t\,dt - \Lambda_t\,dW_t,$$

with $\Lambda_t\,dW_t$ being the price of risk. From the equilibrium solution of the structural model, we obtain

$$-\Lambda_t\,dW_t = -\left[\sigma_{cy}\left(1-\frac{\lambda}{Y_t}\right)+\sigma_c\right]dW_t^c - \sum_{i=1}^{2}\left[\sigma_{\ell_i y}\left(1-\frac{\lambda}{Y_t}\right)\right]dW_t^{\ell_i}, \quad (9)$$

so that the price of risk of unexpected consumption innovations is $[\sigma_{cy}(1-\frac{\lambda}{Y_t})+\sigma_c]$ and the price of risk of the monetary risk factor is $[\sigma_{\ell_1 y}(1-\frac{\lambda}{Y_t})]$. Both prices of risk are state dependent. When the consumption level is close to the habit level H_t, Y_t is low and the local curvature of the utility function is high. The higher implied risk aversion generates a higher price of risk, which affects expected returns independently of interest rate volatility. To see this, let us use Ito's rule to obtain the unexpected innovation in the nominal interest rate $dR_t - E_t dR_t$. We obtain

$$R_t\left(1-\frac{\lambda}{Y_t}\right)\sigma_y\,dW_t^y + R_t\sigma_L\,dW_t^L$$
$$+R_t\,\Xi_t\big[(\Gamma_1+\Gamma_3L_t)(Y_t-\lambda)\sigma_y - (\Gamma_2+\Gamma_3Y_t)L_t\sigma_L\,dW_t^L\big], \quad (10)$$

with

$$\Xi_t = \Gamma_1Y_t + \Gamma_2L_t + \Gamma_3L_tY_t + \Gamma_0.$$

A comparison of (9) and (10) shows that the price of risk is not a constant multiple of interest rate volatility. Similar to Duffee (2002) and Dai and Singleton (2002), we find that such a property is important to simultaneously explain the joint empirical properties of the first and second conditional moments of the term structure and expected bond returns.

III. The Term Structure of Index-Linked Bonds

The term structure of real bonds prices can be obtained by solving $B(t,\tau) = e^{-\rho\tau}E_t[\frac{u_x(X_{t+\tau},H_{t+\tau})\frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t,H_t)\frac{\partial X_t}{\partial C_t}}] = e^{-\rho\tau}E_t[\frac{C_t}{Y_t}\frac{Y_{t+\tau}}{C_{t+\tau}}]$. Since $d(1/C_t) = -(\mu_c - \sigma_c^2)dt - \sigma_c dW_t^c$, it is straightforward to use Lemma 1 to solve for the conditional expectation $E_t[\frac{Y_{t+\tau}}{C_{t+\tau}}]$. Let $\xi_{1t} = \frac{1}{C_t}$ and $\xi_{2t} = Y_t$, and consistent with the notation of Lemma 1 let Θ_y and $\Theta_{1/c}$ be the four-dimensional vector of parameters of the two diffusion processes for dY_t and $d(1/C_t)$.[20] Since the inverse of the consumption is a geometric Brownian motion, $\theta_{\xi_1} = \lambda_{\xi_1} = 0$ and we obtain that $A_0 = A_y = 0$ in Lemma 1. Thus, we obtain the following result.

THEOREM 2 (The Term Structure of Index-Linked Bonds): *The price $B(t,\tau)$ of an index-linked bond with time to maturity τ is equal to*

$$B(t,\tau) = e^{-\rho\tau}\left[A_{Y/c}(\tau;\Theta_Y,\Theta_{1/c}) + A_{1/c}(\tau;\Theta_Y,\Theta_{1/c})\frac{1}{Y_t}\right].$$

Index-linked bonds are affected by monetary innovations through the surplus-consumption ratio $1/Y_t$. The higher the current consumption relative to the current habit stock, the higher $1/Y_t$ and therefore the higher the yield to maturity. Since both monetary and endowment shocks affect Y_t, innovations to both state variables affect the real term structure of interest rates.

Instantaneous Real Interest Rate. The instantaneous real interest rate can be obtained either from taking the limit of the previous result for $\tau \to 0$, or even more simply from the drift of the SDF. Letting $\pi_t = e^{-\rho t}u_x(X_t,H_t)\frac{\partial X_t}{\partial C_t}$ and using Ito's Lemma, one obtains[21]

$$\frac{d\pi_t}{\pi_t} = \mu(\pi,t)\,dt + \sigma(\pi,t)'\,dW_t,$$

with

$$\mu(\pi,t) = -\rho - \mu_c + \sigma_c^2 + k_y\frac{(\theta_y - Y_t)}{Y_t} - \frac{(Y_t - \lambda)}{Y_t}\sigma_{cy}\sigma_c$$

[20] Thus, $\Theta_y = [k_y, \theta_y, \sigma_y, \lambda]$ and $\Theta_{1/c} \equiv [-(\mu_c - \sigma_c^2), 0, -\sigma_c, 0]$.

[21] Observing that $d\pi = d(e^{-\rho t}\frac{Y_t}{C_t})$ and using Ito's Lemma, we have: $\frac{d\pi}{\pi} = -\rho dt + (\frac{dY_t}{Y_t}) - (\frac{dC_t}{C_t}) + (\frac{dC_t}{C_t})^2 - \frac{dY_t}{Y_t}\frac{dC_t}{C_t}$.

and

$$-\sigma(\pi,t)' dW_t = \left[\sigma_{cy}\frac{(Y_t-\lambda)}{Y_t} - \sigma_c\right] dW_t^c + \sigma_{\ell_i y}\frac{(Y_t-\lambda)}{Y_t} dW_t^{\ell_1} + \sigma_{\ell_2 y}\frac{(Y_t-\lambda)}{Y_t} dW_t^{\ell_2}.$$

The real interest rate is given by $r_t = -\mu(\pi,t)$. Thus,

$$r_t = \rho + \mu_c - \sigma_c^2 - k_y\frac{(\theta_y - Y_t)}{Y_t} + \frac{(Y_t-\lambda)}{Y_t}\sigma_{cy}\sigma_c. \tag{11}$$

The first two terms stem from the intertemporal consumption smoothing motive, the third term is the precautionary motive, and the last two terms are due to the presence of the habit. Given the focus of their paper, Campbell and Cochrane (1999) assume constant real interest rates by using a specific value of λ. In our model, this would be equivalent to assuming that $\lambda = -\frac{k\theta}{\sigma_{y_c}\sigma_c}$. Since the focus of our paper is on the dynamics of the term structure of interest rates, we discuss the properties of the model without imposing this restriction.

IV. The Data Set

The empirical results are based on the sample period between January 1960 and December 2000. The data set consists of three main components: interest rate data, price level data, and money supply data. Interest rate data from January 1960 to February 1991 come from the McCulloch and Kwon data set.[22] This database contains end-of-month zero-coupon yields and forward curves based on the McCulloch (1975) methodology from 1 month to 10 years. We extend this data set using the daily GovPX data set, which provides end-of-day prices for all Treasury securities. The data are based on the transactions of the primary dealers through five of the six interdealer brokers for all active and off-the-run U.S. Treasuries. In constructing the zero-coupon yield curve, we follow as closely as possible the methodology of McCulloch (1990) and Kwon (1992). We select the last business day of each month and remove all callable bonds from consideration. The number of Treasury securities in the McCulloch data set increases from slightly over 40 in the 1950s to over 200 in the late 1980s. The average number of Treasury securities in each cross section of our implied spot curve is 134, ranging from 100 to 200.

Inflation data are based on the Consumer Price Index (CPI) for all urban consumers, which is available as of January 1947. The money supply data used in this study are from the official H.6 release of the Federal Reserve Board of Governors. We choose the M2 money stock measure since it includes money market deposit accounts, which can be used for purchasing products and services, and is the closest representation of money in our model. For our purposes,

[22] See McCulloch (1990) and Kwon (1992).

M3 is too broad a measure because it includes instruments that pay significant interest rates and hence cannot be classified as money in our framework. The quarterly per capita consumption series comes from the Citibase data set. From simple summary statistics, we find that the correlation between M2 growth and the yield on the 5-year zero-coupon bond is 15% and the correlation between M2 growth and inflation is 16.8%, lending support to the view that there is an important link between monetary shocks and nominal interest rates.

V. Econometric Methodology

This section uses the restrictions from Propositions 1 and 2 to estimate the structural model. The term structure is not affine as yields are nonlinear functions of a set of underlying factors that follow non-Gaussian diffusions. Methods for estimation and inference that can be applied to continuous-time nonlinear Markov models when data are sampled discretely have been proposed by Lo (1988), Hansen and Scheinkman (1995), Conley et al. (1997), Ait-Sahalia (1996, 1999, 2002), and Stanton (1997), among others.

With discrete-time sampling, Lo (1988) discusses the computation of the likelihood function solving numerically the Fokker-Planck partial differential equation. Since a solution has to be obtained for each maximum-likelihood iteration, Ait-Sahalia (1999) proposes a method to approximate the correct transition function using Hermite polynomials in the context of a maximum-likelihood estimation. Hansen and Scheinkman (1995) construct generalized method of moments (GMM) estimators of the unknown parameter vector using the properties of both the original and reverse-time infinitesimal generator.

With respect to the literature, our econometric task is considerably simplified by the fact that we are in the privileged situation of knowing both the stationary density of the states $z(t)$ in closed form and the conditional moments up to any order. Thus, we can obtain consistent estimators of the unknown parameter vector using GMM. Consistency is achieved for an increasing number of observations ($T \to \infty$), even when these are sampled discretely ($\Delta t > \delta > 0$).

The unobservable states are expressed as functions of the observable economic variables $X(t)$ using the model's solutions. The panel data $X(t)$ consist of both nominal bond yields and macro variables, including real consumption, inflation, and monetary holdings. The vector of moment conditions used to estimate the structural parameters refers to the level of nominal bond yields, moments of the distribution of changes in bond yields, the inflation rate, and the growth in monetary holdings.

A. No-Arbitrage Cross-Sectional Restrictions

The model imposes tight no-arbitrage restrictions across bonds with different maturities. When bond prices are observed with a measurement error, these restrictions take the form of moment conditions. If the rank of the covariance matrix of the observation errors is lower than full rank by at least the number of state variables, then we can use two yields to maturity $y(t, \tau_A)$ to obtain the two (liquidity) states by inverting the yield equations. The third state variable Y_t is directly observable using the result in Proposition 1a. By no-arbitrage the remaining yields $y(t, \tau_B)$ must be equal to a function $\mathcal{F}$ of $y(t, \tau_A)$ plus an observation error.[23] Let

$$h_1\left(y_t^\tau;\theta\right) = y(t, \tau_B) - \mathcal{F}(y(t, \tau_A)).$$

The first set of cross-sectional moment conditions is

$$E\left[h_1\left(y_t^\tau;\theta_0\right)\right] = 0.$$

We select five bond yields to construct the moment condition for h_1.

B. Time-Series Moment Restrictions

B.1. Moments from the Stationary Density

In addition to cross-sectional restrictions on the term structure, we consider a sufficiently large set of moment restrictions generated by both the stationary and conditional density. Let $X(z, t)$ be an observable economic variable that is a function of the state z_t, such as bond yields, and let $\phi(X)$ be a smooth function of X. Since the stationary density $p(z;\theta)$ of the states is known in closed form (Inverse Gamma), moments of X can be easily computed by the integration $E\phi = \int \phi(X)p(z;\theta)\,dz$. The estimation of θ can be posed in a standard GMM framework by setting

$$h_2(X_t;\theta) = [\phi(X_t) - E\phi]. \tag{12}$$

The second set of moment conditions is

$$E[h_2(X_t;\theta_0)] = 0.$$

We choose ϕ and X so that h_2 is a vector of second moments of five bond yields and of the first two moments of M_t, P_t, and C_t. The total number of moment restrictions is therefore $5 + 6 = 11$.

[23] Clearly, the choice of which bonds are observed with no error is arbitrary (i.e., the generalized inverse of the covariance matrix is not unique). Thus, we select the instruments that imply the lowest pricing errors for the remaining bonds. We find that this is achieved when we select the 3-month and 2-year yield to maturity. However, the empirical results are not very sensitive to this choice as long as they sufficiently span the maturity structure.

Similarly, when the inverse function of the instrument is not unique, we choose the state vector that minimizes the bond pricing errors.

B.2. Conditional Moment Restrictions

Let $\phi(z)$ be a smooth function of the state vector $z_t = [Y_t, \ell_t]$. Consider a Taylor expansion of $\phi(z_{t+1})$ at z_t. Taking the conditional expected value, we have

$$E_t\phi(z_{t+1}) = \phi(z_t) + \sum_{i=1}^{J} \frac{1}{i!}\frac{\partial^i}{\partial z^i}\phi(z_t)E_t[z_{t+1} - z_t]^i + o\big(E[z_{t+1} - z_t]^i\big).$$

The moments $E_t\,[z_{t+1}]^i$ can be obtained using the results of the following proposition, so that the conditional expectations of $E_t\phi(z_{t+1})$ can be obtained up to any desired degree of approximation.

PROPOSITION 3: *Given a linear stochastic process z_t satisfying assumption (A2) with entrance boundary λ, the conditional second moment is equal to*[24]

$$E_t\big(z_T^2\big) = e^{-(2k-\sigma^2)T}z_0^2 + 2(k\theta - \lambda\sigma^2)\left[\frac{\theta}{(2k-\theta)} + \frac{e^{-kT}(z_0-\theta)}{(k-\theta)}\right].$$

All other moments can be obtained recursively by integrating the differential equation

$$\frac{d}{dt}E_t\big(z_T^n\big) = E_t\big(z_T^{n-2}\big)\left[-nk + \frac{n(n-1)}{2}\sigma^2\right] + E_t\big(z_T^{n-1}\big)[nk\theta].$$

Note that in order for the process to admit a finite unconditional moment of order n, the parameters must satisfy the condition $[-2k + (n-1)\sigma^2] < 0$. This restriction becomes progressively tighter as the order n increases.

For the estimation, we choose ϕ to be $[y(t+1,\tau), y^2(t+1,\tau)]$, so that the parameters are estimated using the first two conditional moments of the observable process. Let

$$h_3(X_t,\theta) = \big[y_{t+\Delta}^{\tau} - E\big(y_{t+\Delta}^{\tau} \mid X_t\big)\big] \otimes \xi(X_t)$$

and

$$h_4(X_t,\theta) = \big[\big(y_{t+\Delta}^{\tau}\big)^2 - E\big(\big(y_{t+\Delta}^{\tau}\big)^2 \mid X_t\big)\big] \otimes \xi(X_t).$$

We select the lagged values of consumption and money growth to build a set of instruments $\xi(X_t)$. Thus, since the number of independent bond yields is five, the total number of restrictions from this set of moments is $5 \times 2 \times 2 = 20$.

B.3. The State Variables

An explicit characterization of $Y(t)$, the inverse surplus-consumption ratio, as a function of the accumulated consumption and monetary shocks is obtained in Proposition 1a as a solution of (*A2*): $Y(t) = \lambda + \omega_t^{-1}[(Y_0 - \lambda) + k(\theta -$

[24] Clearly, the process ℓ_t is a special case of Y_t with boundary $\lambda = 0$.

$\lambda)\int_0^t \omega_s\, ds]$, with $\omega_t = \exp\{(k + \frac{1}{2}\sigma_y^2)t + \sigma_y(W_t - W_0)\}$, where $\sigma_y\, dW_t = \sigma_y(\frac{dX}{X} - E\frac{dX}{X}) = \sigma_y((1-\gamma)\sigma_y(\frac{dC_t}{C_t} - \mu_c) + \gamma(\frac{dm_t}{m_t} - \mu_m))$. The other unobservable state variables are ℓ_{it}. They can be obtained by inverting two of the measurement equations to express ℓ_{it} as a function of the remaining vector of observable economic variables and θ, $\ell_{it} = \mathcal{F}(X_t, \theta)$. This approach is also used by Chen and Scott (1993) and Duffie and Singleton (1997) for the swap curve. Once we substitute these restrictions on Y_t and ℓ_{it} back into $h_t(X_t, \theta)$, all moment conditions depend exclusively on observable economic variables and structural parameters.

C. Estimation

We merge the no-arbitrage cross-sectional restrictions and the moment conditions from both the stationary and conditional distributions in a vector $h_t(X_t; \theta)$, with $h_t = [h_1, h_2, \ldots, h_4]'$, so that

$$E\left[h_t(X_t;\theta)\right] = 0. \tag{13}$$

The total number of moment conditions is therefore $5 + 11 + 20 = 36$. Since the model has 19 parameters, the model is overidentified and the number of degrees of freedom is 17. One can obtain a consistent estimator of the vector of structural parameters by minimizing the quadratic criterion $J_T(X_t; \theta)$ with respect to θ, based on the sample counterpart of the previous moments

$$J_T(X_t;\theta) = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_t(X_t;\theta)\right]' W_T^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_t(X_t;\theta)\right]. \tag{14}$$

Under the null hypothesis that the model is correctly specified, $J_T(X_t; \theta) \to \chi^2_{(17)}$. This asymptotic distribution can be used to construct test statistics for the overidentifying restrictions of the model. It is important to notice that to obtain consistent estimators for θ depends on the number of observations $T \to \infty$, not that the sampling frequency converges to zero. This property differs from other estimation techniques for nonlinear continuous-time models. The reason is simple. Even if the model is nonlinear, we can identify the unconditional and conditional moment conditions even for a strictly positive sampling frequency $\Delta t > 0$. Thus, for our model consistency in the estimation does not require continuously observed sample processes. The weighting matrix W_T is the Newey–West heteroskedasticity and autocorrelation-consistent estimator of the covariance matrix, namely, $W_T = \Gamma_0 + \sum_{i=1}^{q}(1 - \frac{i}{q+1})(\Gamma_i + \Gamma_i')$, with $\Gamma_i = \sum_{t=i+1}^{T}(h_t - h)(h_{t-i} - h)'$ and $h = \frac{1}{T}\sum_{t=1}^{T} h_t$.

D. Small-Sample Properties

Since the model is nonlinear and most of the test statistics rely on asymptotic results, we check their small-sample properties. To do so, we select a parameter configuration θ_0 and simulate the data-generating process

$\{\tilde{X}_{j1}(\theta_0), \ldots, \tilde{X}_{jT}(\theta_0)\}_{j=1}^{N}$. The assumed parameter configuration corresponds to the sample estimates of the model. Then, we estimate $\hat{\theta}$ using the moment conditions $Eh_t(\tilde{X}_t, \theta) = 0$ computed for each simulation by minimizing the quadratic criterion (14). We obtain $\{\hat{\theta}_1, \ldots, \hat{\theta}_N\}$. Under the null hypothesis that the estimators are unbiased in small samples, $\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\hat{\theta}_j \to \theta_0$ for finite T. Then, we compute the empirical rejection ratio of the overidentifying no-arbitrage restrictions. We find that the empirical rejection frequency is 6.1%, as opposed to the theoretical value of 5%. The test slightly overrejects the model in small samples: The asymptotic tests tend to be biased on the conservative side when used in small samples.

VI. Empirical Results

This section summarizes the results based on the estimation of a three-factor model.

A. Joint Test of the Overidentifying Restrictions

Table 1 reports estimates of the parameters and their corresponding standard errors.[25] It is easy to verify that $2k_y > \sigma_y^2$, $\theta_y > \lambda > 1$, and $k_y > 0$, so that Y_t satisfies the conditions (*C1*) and (*C2*) for Y_t to be stationary and square-integrable. The equivalent conditions for the liquidity factors ℓ_{1t} and ℓ_{2t} are also satisfied. Moreover, the two conditions for the existence of a monetary equilibrium require $-k_y - k_{\ell_i} - (\sigma_{cy}\sigma_{\ell_i}\rho + \sigma_{\ell_i y}\sigma_{\ell_i}) < 0$ for $i = 1, 2$. These conditions are satisfied and, at the estimated parameter values, are equal to -0.35 and -0.22, respectively. Finally, a sufficient condition for $\mathrm{cov}(dH, dC) > 0$ is that $\sigma_y \geq \frac{v\sigma_c^2 - v\gamma c^* + \lambda\psi_c v^2}{\sigma_c^2 - \gamma c^* - v\psi_c}$, where $\psi_c = \sigma_{yc}\sigma_c - \sigma_{y\ell_i}\sigma_c\rho_{\ell_i,c}$ and $v = \frac{\sigma_y\psi_c + \sigma_c^2 - \gamma c^*}{2\lambda\psi_c}$ with $c^* = \max_i \sigma_{\ell_i}\sigma_c\rho_{c,\ell_i}$. The condition is satisfied.

To assess the model's overall goodness of fit we first run a joint GMM test on the model's overidentifying restrictions. The test is based on all moment conditions. Under the null hypothesis that the model is correctly specified, the maximum $J_T(X_t, \hat{\theta})$ is asymptotically chi-squared distributed with 17 degrees of freedom.[26]

$$J_T(\hat{Y}, \hat{g}, \hat{\theta}) = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h(X_t, \hat{\theta})\right]' W_T^{-1} \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h(X_t; \hat{\theta})\right] \sim \chi^2_{(17)}.$$

The model-implied p-value, based on all overidentifying moment conditions, is 36%. The model is not rejected.

Campbell and Cochrane (1999) and Menzly et al. (2004) choose a parameter configuration to match a set of unconditional moments of real equity returns. Some of these parameter values are different from our estimates. For instance,

[25] The asymptotic covariance matrix of the parameters is based on the outer product of the Jacobian of the log-likelihood function.

[26] See Hansen (1982).

Table I
Parameter Estimates

This table presents the estimates of the structural parameters. The estimation is based on the asset pricing restrictions for eight nominal bonds with maturities ranging from 3 months to 10 years, as well as the processes for the money supply M2, inflation, and the habit. The estimated model has three factors. The inverse consumption surplus factor follows

$$dY_t = k_Y(\theta_Y - Y_t)\,dt - (Y_t - \lambda)\left[\sigma_{cy}\,dW_t^c + \boldsymbol{\sigma}_{l\,y}\,dW_t^l\right].$$

We assume two liquidity shocks ℓ_{it} affecting the money supply, which follows $d\,\ell_{it} = k_{l_i}(\theta_{l_i} - \ell_{it})\,dt + \sigma_{\ell_i}\ell_{it}\,dW_t^{\ell_i},\ i = 1, 2$. The Brownian motions W_t^l and W_t^c are assumed to be correlated. In parentheses we report the p-values of the likelihood ratio test.

ρ	k_Y	θ_Y	σ_{cy}	λ
0.0150	0.0218	11.0012	0.2580	10.4007
(0.01)	(0.02)	(0.01)	(0.01)	(0.03)
$\sigma_{l_1 y}$	$\sigma_{l_2 y}$	$\rho_{l_1 c}$	$\rho_{l_2 c}$	σ_c
0.0631	0.0511	0.2380	0.3185	0.0157
(0.01)	(0.01)	(0.01)	(0.02)	(0.02)
k_{l_1}	θ_{l_1}	σ_{l_1}	k_{l_2}	θ_{l_2}
0.3662	0.8218	0.0109	0.2401	3.1978
(0.02)	(0.01)	(0.01)	(0.01)	(0.01)
σ_{l_2}	μ_c	μ_m	γ	
0.0106	0.0135	0.0432	0.5133	
(0.01)	(0.03)	(0.01)	(0.02)	
		Test of Overidentifying Restrictions		
		$J_T = 18.38,\ pval = 0.36,\ df = 17$		

Menzly et al. (2004) choose $k_y = 0.16$ and $\sigma_y^2 = 0.62$, which are both higher than the values that we estimate. On the one hand, since they calibrate a number of moments that is smaller than the number of their free parameters, the difference in parameter values is not surprising. Different parameter configurations imply the same moments in their exercise. Moreover, since the stationary distribution p of a diffusion process satisfies $\frac{d}{dx}[p(x)\sigma^2(x)] = 2\mu(x)p(x)$, the drift and volatility parameters μ and σ are identified only up to a scale factor. That is, Menzly et al. (2004) could scale the choice of k and σ_y^2 by a factor of 10 without affecting the unconditional moments of the observable processes.[27] Clearly, however, even if the two parameter configurations produce similar *unconditional* moments, they generally imply different *conditional* moments and since bond prices are conditional moments of the SDF, this difference plays a crucial role in our study. For this reason, in our empirical analysis and

[27] To see this, note that the statistical distribution of Y_t is an Inverse Gamma with parameters $-2(\frac{k}{\sigma^2} - 1)$ and $2\frac{k\theta}{\sigma^2}$.

tests we pay special attention to the conditional moment properties of interest rates.

B. Nominal Yield Curve

The average fitting errors of the yield curve range between 27 and 60 basis points, see Table II, Panel A. Since in the estimation the same three factors are required to fit the consumption process, the inflation rate, the money supply growth rate, and the second moments of the yields, we also report the results when the model is estimated exclusively using yield curve moment conditions. The overall mean absolute error of the yield curve drops to 21 basis points, see Table II, Panel B.

We find that the yield curve is higher during low surplus-consumption ratio (high habit stock) periods, that is, recessions. In these states, investors' marginal utility is high, implying a higher consumption demand. Figure 1 illustrates the relationship between the yield curve and the surplus-consumption ratio for different levels of the money growth rate. This relationship is monotone both at high and low monetary growth levels. The closer the consumption to the habit, the higher the level of the yield curve. A one-standard deviation change in the surplus-consumption ratio induces a 60-basis point change in the level of the yield curve. This is about the same average yield difference between a 3- and 8-year bond. Moreover, we find that the yield curve is steeper during periods of high monetary growth. For instance, when the money growth rate is one standard deviation above its long-run mean, the slope of the yield curve is 370 basis points. The average slope is 40 basis points. This can be due to the following. First, a high money growth rate anticipates high future inflation rates, which translates to higher current long-term yields. Second, since real monetary holdings are mean reverting, high current values, which are expected to revert to the long-term mean, reduce the surplus-consumption ratio and therefore increase current long-term yields.

C. Real Interest Rates and Consumption

We compare the behavior of the model-implied and empirical real yields using index-linked bonds. We use the approach suggested by Green and Odegaard (1997) and Buraschi and Jiltsov (2005) to control for the tax implications of the inflation adjustment of the principal amount. We focus this part of the analysis on the sample period after January 29, 1997, which is the date of the first Tips issue. The model-implied mean and standard deviation of the 3-month real yield are equal to 2.13% and 1.13%, respectively. The corresponding empirical values are equal to 2.88% and 0.47%. The model can reproduce the first moment of the 3-month real interest rate, but it clearly underestimates its volatility. The p-value of a GMM test for the joint null hypothesis that the first two moments are correctly specified is 5%. However, the null hypothesis that the second moment of the 3-month real yield is correctly specified is rejected.

Table II

Goodness of Fit by Maturity

This table presents fitting errors for the model measured in basis points. The fitting errors are defined as the difference between the model-generated nominal spot rate and the observed nominal rate during the sample period. The maturity of the bonds ranges between 3 months and 10 years. Panel A reports the estimation results when all moment conditions are used. These moments include asset pricing restrictions as well as restrictions from the money, inflation, and habit process. Panel B reports the fitting errors of an estimation based exclusively on the yield curve restrictions of the model.

	Min Error	Max Error	Mean Abs Error	Median Abs Error
Panel A: Term Structure Plus Macro Variable Fit				
3 months	−411.7	254.9	61.6	33.3
6 months	−473.0	173.1	78.9	58.8
1 year	−377.4	246.3	54.4	30.7
2 years	−278.9	300.8	47.8	27.1
3 years	−219.7	314.8	46.6	31.2
5 years	−182.4	321.2	41.6	26.8
7 years	−155.2	314.0	45.6	32.1
10 years	−185.5	275.5	71.2	62.5
Panel B: Term Structure Fit Only				
3 months	−36.4	42.7	12.8	11.7
6 months	−80.6	−1.7	37.6	38.0
1 year	−72.4	94.4	19.7	14.6
2 years	−83.1	130.0	32.8	28.2
3 years	−68.4	117.3	31.4	26.1
5 years	−61.7	70.5	17.6	13.1
7 years	−47.0	23.1	10.1	8.7
10 years	−54.1	−26.8	38.4	38.3

The model fares better with long-term real yields. The model-implied mean and standard deviation of the 10-year real yield are equal to 2.51% and 0.44%, respectively. The corresponding empirical values are equal to 2.84% and 0.34%. The p-value of a GMM test on the joint null hypothesis that the first two moments are correctly specified is 12%.

The model-implied mean and standard deviation of the consumption growth rate are equal to 2.4% and 1.7%, respectively. The corresponding empirical values are equal to 1.9% and 1.9%. The model can fit bond prices without an excessively volatile consumption process. We run a joint test for the null hypothesis that the model-implied first two moments correspond to the empirical values and find that the null hypothesis is not rejected with a p-value of 0.11.

We also compute the model forecast errors for the real consumption growth rate. The mean absolute deviations are 38, 72, and 119 basis points for 3-month, 6-month, and 1-year horizons, respectively. These compare to forecasts errors equal to 34, 60, and 109 basis points obtained using an ARMA(1,1) model.

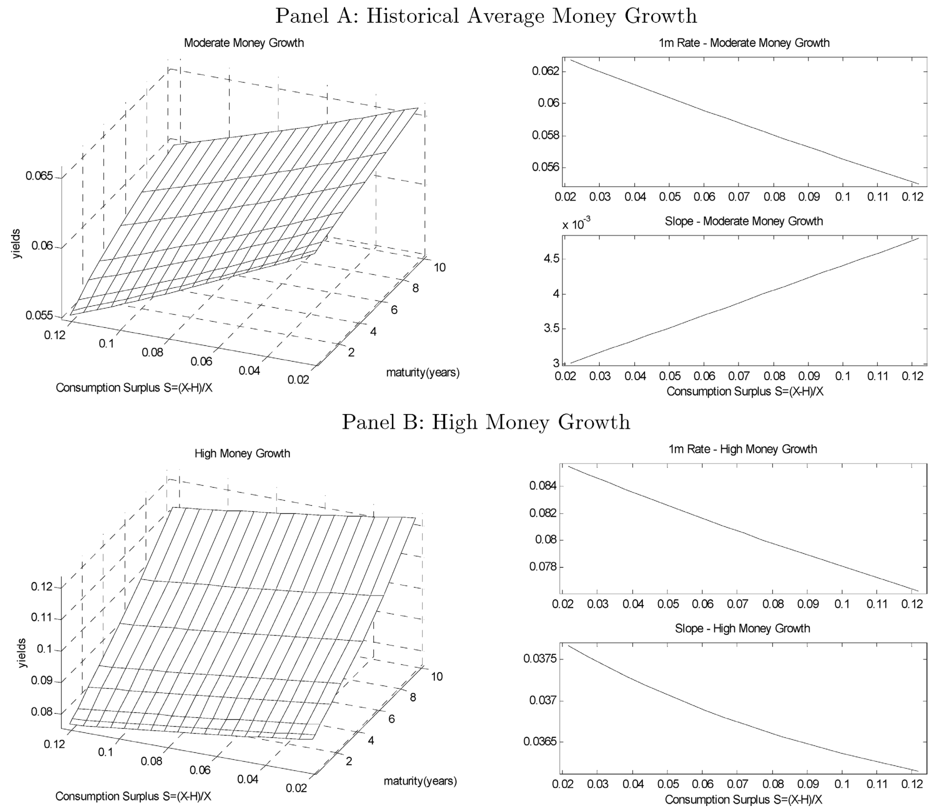

Figure 1. Sensitivity of bond yields to habit level. This figure illustrates the bond yields sensitivity to the inverse of the surplus ratio, that is Y_t. We consider two regimes, moderate money growth (historical mean value, Panel A) and high money growth (two standard deviations higher than the historical mean, Panel B).

D. Inflation

Figure 2 plots the model-implied and realized inflation rate. The model-implied mean and standard deviation of the inflation rate are equal to 4.03% and 2.33%, respectively. The corresponding empirical values are equal to 4.70% and 3.23%. The p-value of a GMM test for the null hypothesis that the model-implied first two moments are equal to their empirical counterparts is 0.25. The model-implied inflation forecasting errors are 29, 57, and 115 basis points a 3-, 6-, and 12-month horizons, respectively. For the sake of comparison, we compute the forecasting errors implied by an exogenous ARMA(1,1) specification and find that the corresponding values are 28, 53, and 110 basis points.

We run an orthogonality test on the model prediction errors to test the inflation process's goodness of fit. Let $E_t(.\ \pi_{t+1} \mid I_t)$ be the model-implied expected inflation rate. If the model is correctly specified, the prediction errors should be orthogonal to any function of x_t that is measurable with respect to I_t. If the model is not correctly specified, some function of the explanatory variable $\phi(x_t)$

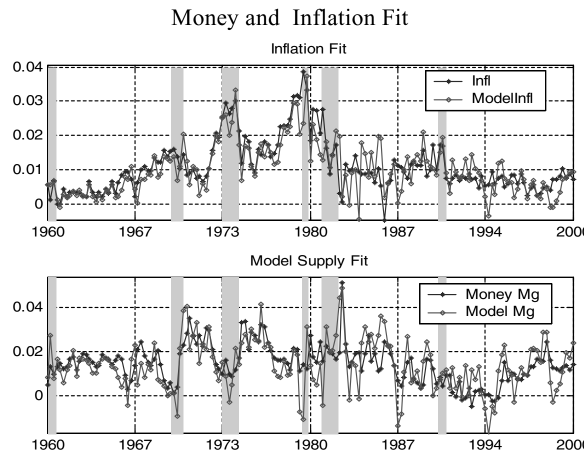

Figure 2. Quality of fit for macro variables. This figure plots the model-implied time series of inflation and money supply together with their empirical counterparts. Gray boxes on the graph show the periods of the U.S. recessions compiled and reported by NBER.

would improve the model forecasts, that is, $E_t(\pi_{t+1} \mid I_t) + \theta'\phi(x_t)$. Consider the inflation forecast error $u_{t+1} = \pi_{t+1} - E_t(\pi_{t+1} \mid I_t) - \theta'\phi(x_t)$, and the null hypothesis $H_0 : \theta = 0$. Define $h(x_t, \theta)$ as a function of the prediction errors

$$h(x_t, \theta) = \begin{bmatrix} u_{t+1}(\theta) \\ u_{t+1}(\theta) \otimes [\xi(x_t)] \end{bmatrix}.$$

Under the null hypothesis that the model is correctly specified $\theta_0 = 0$, and $Eh(x_t, \theta_0) = 0$. Using a standard GMM approach we observe that under the null hypothesis, the following d_T statistics are distributed as a χ^2

$$d_T = T \cdot \left[h_T(x_t, \theta_0)' W_T^{-1} h_T(x_t, \theta_0) - h_T(x_t, \theta)' W_T^{-1} h_T(x_t, \theta)\right],$$

where θ_0 is the parameter restricted to zero and θ is the unrestricted parameter, with $h_T(x_t, \theta) = \sum_{t=1}^{T} h_t(x_t, \theta)$.

The results are reported in Table III. We find that the p-value of the orthogonality tests is 0.066, 0.156, and 0.374 at the 3-month, 6-month, and 1-year horizons, respectively. Thus, we do not reject the null hypothesis that the inflation prediction errors are orthogonal to $\phi(x_t)$.

E. The Term Structure of Second Moments

We now explore the extent to which the model can reproduce the time variation of the conditional second moments of the term structure of interest rates once we fix the parameters at their estimated values.[28] Let ϕ_t^n be the

[28] Dai and Singleton (2000, 2002) and Backus, Telmer, and Wu (1999) document the trade-off of traditional models in explaining the first and second moments of interest rates.

Table III

Orthogonality Tests

The table shows the results of the orthogonality tests of the prediction errors of growth rates in consumption, the price index, and monetary holdings. We test the null hypothesis $H_0 : \theta = 0$ in a GMM framework with the following moment restrictions:

$$\begin{bmatrix} u_{t+12}(\theta) \\ u_{t+12}(\theta) \otimes [\xi(x_t)] \end{bmatrix}.$$

In the case of the consumption equation, let $c_{t+1} = \ln(\frac{C_{t+1}}{C_t})$. The prediction errors are defined as

$$u_{t+1} = c_{t+1} - E_t\left[c_{t+1} \mid I_t\right] - \theta' \phi(x_t).$$

We test the null hypothesis using the following statistics d_T :

$$d_T = T \cdot \left[h_T\left(x_t, \theta\left(H_0\right)\right)' W_T^{-1} h_T\left(x_t, \theta\left(H_0\right)\right) - h_T\left(x_t, \theta^*\right)' W_T^{-1} h_T\left(x_t, \theta^*\right)\right],$$

which is distributed χ^2 under the null hypothesis. We consider the following set of lagged explanatory variables: $\phi(x_t) = const, c_{t-1}, c_{t-1}^2$. We report the value of the GMMd_T statistics with their corresponding χ^2 p-values in parentheses.
Panels B and C present the results of the same orthogonality tests for inflation and money growth. We report the value of test statistic d_T for three time horizons, 3 months, 6 months, and 1 year.

	3 months	6 months	1 year
Panel A: Orthogonality Test of Consumption			
d_T	3.319	3.173	4.922
	(0.345)	(0.366)	(0.178)
Panel B: Orthogonality Test of Inflation			
d_T	7.180	5.220	3.115
	(0.066)	(0.156)	(0.374)
Panel C: Orthogonality Test of Money			
d_T	39.257	11.538	6.098
	(0.000)	(0.009)	(0.107)

model-implied conditional second moment, which is obtained by solving $E_t\,[\Delta y^n_{t+\Delta t}]^2$ in closed form using the model's structural restrictions. We run a regression[29]

$$\left(\Delta y^n_{t+\Delta t}\right)^2 = \alpha + \beta \times \phi^n_t + \varepsilon_{t+\Delta t}.$$

We test the null hypothesis that $H_0 : \alpha = 0$ and $H_0 : \beta = 1$. The results are summarized in Table IV, Panel B. We fail to reject the null hypothesis that $H_0 : \alpha = 0$ for maturities above 3 years. Moreover, we do not reject the null hypothesis that $H_0 : \beta = 1$ for any maturity. The R^2s of the regressions range

[29] The spirit of this regression analysis is similar to the one outlined in Duarte (2004). Our regression, however, applies to the volatility of the *changes* in bond yields, as opposed to that of the *level* of bond yields.

Table IV

Conditional Volatility

In Panel A, we test the correct specification of the model-implied conditional volatility. Given the closed-form model solution for the second noncentral moment of yield changes, denoted as $\mathcal{M}^V(Y_t, g_{it}, \theta)$, we construct a GMM test based on the following moment conditions:

$$h_{t+\Delta t} = \left(\Delta y_t^\tau\right)^2 - \mathcal{M}^V\left(Y_t, g_{it}, \theta\right),$$

from which we construct the following χ^2 statistics

$$d_T = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right]' W_T^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right].$$

In Panel B, we show how well the model-implied conditional volatility of yield changes predicts the future realized volatility. We solve for the second moment implied by the model and run the following regression:

$$\left(\Delta y_{t+\Delta t}^n - E_t\left[\Delta y_{t+\Delta t}^n\right]\right)^2 = \alpha + \beta \times \Phi_t + \varepsilon_{t+\Delta t},$$

where Φ_t is the *Model-Implied Conditional Second Moment of* Δy. We test the null hypothesis that $H_0 : \alpha = 0$ and $H_0 : \beta = 1$. The p-values for $H_0 : \alpha = 0$ are given in parentheses under the respective values for α. The p-values for $H_0 : \beta = 1$ are given in the last row before the R^2.

	3 months	6 months	1 year	2 years	3 years	5 years	7 years	10 years
				Panel A: An Asymptoic GMM Test				
d_T	0.808	1.249	1.601	1.749	1.921	2.078	2.005	1.764
p-value	0.369	0.264	0.206	0.186	0.166	0.149	0.157	0.184
	Joint Test (All Maturities)					p-value = 0.21		
				Panel B: Predictive Power				
α	−0.003	−0.003	−0.002	−0.001	−0.001	−0.000	−0.000	0.000
β	1.202	1.142	1.046	1.004	0.971	0.896	0.828	0.771
R^2	0.259	0.237	0.204	0.189	0.182	0.158	0.133	0.110
p-value, $H0 : \beta = 1$	0.209	0.382	0.780	0.980	0.858	0.526	0.303	0.186
p-value, $H0 : \alpha = 0$	0.002	0.003	0.011	0.035	0.052	0.215	0.594	0.695
	Joint Test (All Maturities) $H0$:$\beta = 1$					p-value = 0.42		
	Joint Test (All Maturities) $H0 : \alpha = 0$					p-value =0.07		

from 11% for the 10-year yield to maturity to 26% for the 3-month yield to maturity. These results can be compared with those of Duarte (2004), who runs a similar test both on a three-factor CIR model and on his own model with a flexible specification for the price of risk. Based on the 1983 to 1998 sample period,[30] he rejects the null hypothesis that $H_0 : \beta = 1$ and reports an R^2 that ranges between 7% and 15% for the CIR model. The persistence of the habit stock as a state variable clearly helps explain the yield change volatility at the

[30] This time span does not include the period of high interest rate volatility, which is more difficult to predict.

short end of the term structure as the R^2s are larger than those reported in Duarte (2004). We find this persistence less helpful, however, in the long end of the term structure.

We also compute an asymptotic GMM test of the model's ability to reproduce the conditional second moments of bond yields. For each maturity n, we construct the test statistic $d_T = [\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}]' W_T^{-1} [\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}]$, with $h_{t+\Delta t} = (\Delta y^n_{t+\Delta t})^2 - \phi^n_t$. This statistic is distributed as a chi-square with one degree of freedom. The results are reported in Table IV, Panel A. The p-value of the test ranges from 37% for the 3-month yield to 18% for the 10-year yield to maturity. A joint test based on all maturities gives a p-value equal to 21%. We conclude that the model can produce sufficient time variation in the conditional second moments.

VII. Nonlinear Interest Rates

Our model implies a nonlinear short-term interest rate process. To investigate the nonlinearity of interest rates, Conley, Hansen et al. (1997) study the pull function $\wp(r)$, which is defined as the conditional probability that the process r_t reaches the value $r + \varepsilon$ before $r - \varepsilon$, if initialized at $r_0 = r$. Formally, let T_ε be the local hitting time $T_\varepsilon = \inf(t \geq 0; r_t = r + \varepsilon)$. Then $\wp(r)$ is defined as $\wp(r) = \Pr\{T_{r+\varepsilon} < T_{r-\varepsilon} | r_0 = r\}$. In practice, $\wp(r)$ is computed as

$$\wp(r) = \frac{\mathcal{S}(r) - \mathcal{S}(r-\varepsilon)}{\mathcal{S}(r+\varepsilon) - \mathcal{S}(r-\varepsilon)},$$

where $\mathcal{S}(y) = \int^y s(x)dx$, with $s(x)$ being the scale function of the interest rate process, that is $s(x) \equiv \exp[-\int^x \frac{2\mu_r(v)}{\sigma_r^2(v)} dv]$. Solving the previous equation, $\wp(r) = \frac{1}{2} + \frac{\mu_r(r)}{2\sigma_r^2(r)}\varepsilon + o(\varepsilon)$, where $\mu_r(r)$ and $\sigma_r(r)$ are the drift and local volatility of the interest rate process. Thus, $\wp(r)$ has a simple and intuitive interpretation: It is a conditional measure of mean reversion (scaled by twice the local variance). Conley et al. (1997) estimate $\wp(r)$ using the properties of subordinated diffusions for alternative interest rate specifications and suggest a test statistic to compare the model-implied $\wp(r)$ with its sample counterpart. They find statistical evidence of nonlinearity in $\mu_r(r)$. In what follows, we use their methodology to investigate the difference between the model-implied pull function, computed at the estimated structural parameter values, and its empirical counterpart, estimated using semiparametric methods.

To estimate the model-free $\wp(r)$, we assume a flexible polynomial specification of the local volatility $\sigma(r) = \sum_{i=0}^{\gamma} \sigma_i r^i$ and follow Conley et al. (1997) to estimate the drift $\mu(r)$. The results are shown in Figure 3. The pull function is positive for $r_t \leq 5.4\%$ and it goes above 10 for interest rate values below 4%. The model-implied $\wp(r)$ observes similar behavior and it remains inside the confidence

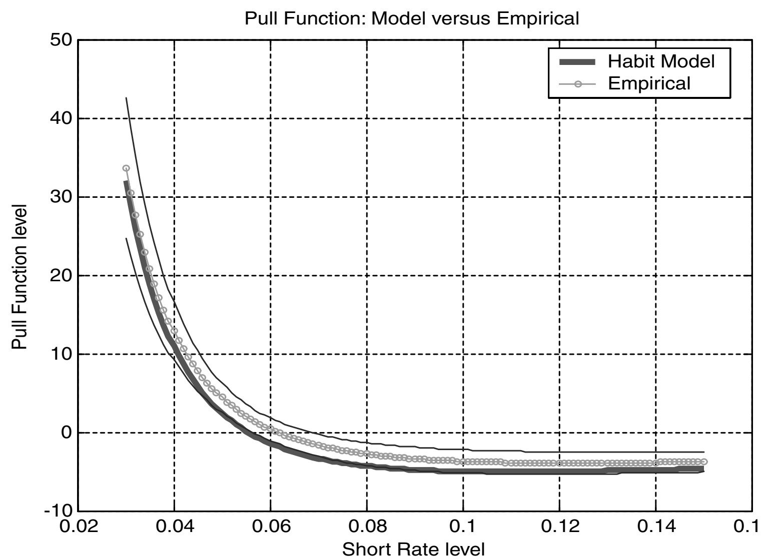

Figure 3. Pull function. The plots shows the empirical pull function based on the flexible semi-parametric approach by Conley et al. (1997), which uses a flexible parameterization of the local volatility $\sigma(r_t) = \sum_{i=0}^{\gamma} \sigma_i r_t^i$, and the habit model-implied pull function. The two thin solid lines are the 95% confidence bounds around the empirical functions.

bounds for most of the support. We run a test of the null hypothesis that the pull function implied by the habit model is equal to the empirical pull function and find that the p-value is equal to 0.43. We do not reject the null hypothesis that the model is correctly specified. Second, we impose the restriction that the interest rate drift $\mu_r(r)$ is linear and recompute the pull function. The p-value is now equal to 0.02 and the pull function is outside the confidence bounds for large sections of the support. We conclude that the type of nonlinearity in interest rates implied by the model is consistent with the data in our sample period.

VIII. The Lead-Lag Relation between Interest Rates, Consumption, and Money

The lead-lag correlations among real interest rates, consumption, and output are important properties in the real business cycle literature. Fiorito and Kollintsas (1994) and Chari et al. (1995) report that real and nominal interest rates are positively (negatively) correlated with past (future) detrended output. Wachter (2005) focuses on a real interest rate model and finds that future real interest rates are predicted by past average consumption. King and Watson (1996) and Boldrin et al. (2001) view the lead-lag relationships among consumption, output, and interest rates as an important challenge to standard models with time-separable preferences, such as that proposed by Cos, Ingersoll, and

Ross (1985b). They argue that the original model should be extended to include more general preferences and/or monetary policy shocks. The empirical evidence reported in these studies is consistent with the predictions of our model. To see this, observe that equation (11) implies that $\partial r_t/\partial Y_t > 0$. The relationship between Y_t and the nominal interest rate R_t given by equation (8) is more complicated since it involves the indirect effect of Y_t on the inflation rate. Nonetheless, it is possible to show that $\partial r_t/\partial Y_t > 0$. This implies that an increase in lagged consumption increases the habit stock, thereby reducing S_t and increasing Y_t, with the resulting effect of increasing interest rates. Since money reduces transaction costs, lagged positive innovations in the money supply have a similar effect.

We investigate these relationships by regressing the short-term real and nominal interest rates on the Y_t process. The exact solution for Y_t is given in Proposition 1 as $Y(t) = \lambda + (Y_0 - \lambda)e^{\phi t}\ \exp\{-\sigma_y(W_t - W_0)\} + k(\theta - \lambda)\int_0^t e^{\phi(t-s)}\ \exp\{-\sigma_y(W_t - W_s)\}\,ds$, where $\phi = \exp(-(k_y + \frac{1}{2}\sigma_y^2))$. We construct an empirical time series for Y_t as a function of lagged observed innovations in consumption and monetary holdings by discretizing the previous solution. This gives us $\tilde{Y}_t$. We then study the following regressions:

$$r_{t+1} = \alpha_1 - \beta_1\tilde{Y}_t(\phi, \gamma) + \varepsilon_{t+1}$$

$$R_{t+1} = \alpha_2 - \beta_2\tilde{Y}_t(\phi, \gamma) + \eta_{t+1}.$$

An empirical prediction of the model is that $H_0 : \beta_1 < 0$ and $H_0 : \beta_2 < 0$. We present the results for a full grid of parameter configurations (ϕ, γ), to test their robustness against a range of parameter values, and for the parameter values obtained by estimating the structural model.

Real Interest Rate. The regression results are given in Table V, Panel A. At the estimated parameter values of the structural economy, $k_y = 0.02$ and $\sigma_y = 0.027$, hence $\phi = 0.96$ and $\gamma = 0.51$. With this parameter configuration, we obtain $\beta_1 = -0.17$ and $R^2 = 15\%$. The slope coefficient is negative and significantly different from zero as predicted by the model. The result is robust to any parameter configuration (ϕ, γ) such that $\phi > 0.90$. Moreover, we find that the result is not very sensitive to γ, both in terms of R^2 and statistical significance of the slope coefficient.

Nominal Interest Rate. The regression results for the nominal interest rates are given in Table V, Panel B. At the estimated parameter values of the structural economy, the slope coefficient is -0.32 with a t-statistic equal to -7.5. The R^2 is 34%. Can we use a real term structure model, similar to that in Cox, Ingersoll, and Ross (1985b) and Wachter (2005) to explain the dynamics of the nominal yield curve? The simple answer is "no." Nominal interest rates are strongly influenced by monetary fundamentals. To see this, notice that when $\gamma = 0$ (see first column in Table V, Panels A and B), the $\tilde{Y}_t$ process has very limited power to explain the interest rate process; the R^2 hardly reaches 1%. For this parameter value, money does not provide any transaction service and is not held in equilibrium. The model economy is real. When the real monetary

Table V

Lead-Lag Relation between Interest Rates and Habit

We run the following regressions:

$$\text{Panel A}: r_{t+1} = \alpha_1 + \beta_1 \tilde{Y}_t(\phi, \gamma, n) + \varepsilon_{t+1}$$

$$\text{Panel B}: R_{t+1} = a_1 + b_1 \tilde{Y}_t(\phi, \gamma, n) + \varepsilon_{t+1},$$

where r_{t+1} and r_{t+1} are the realized real and nominal interest rates. $\tilde{Y}_t$ is the discretized time-series process of Y_t obtained from the solution in Proposition 1. $\phi = \exp[-(k + \frac{1}{2}\sigma_y^2)]$.

$\phi\backslash\gamma$	0.000	0.050	0.100	0.400	0.600	0.800	1.000
Panel A: Predictability of **Real** Interest Rate by Money-Adjusted Habit: $R^2(\phi, \gamma)$							
0.200	0.001	0.003	0.005	0.018	0.026	0.032	0.037
0.500	0.001	0.003	0.005	0.018	0.024	0.029	0.032
0.700	0.000	0.000	0.001	0.007	0.011	0.013	0.015
0.900	0.019	0.019	0.019	0.015	0.013	0.012	0.011
0.950	0.087	0.098	0.105	0.103	0.095	0.089	0.084
0.980	0.194	0.219	0.233	0.223	0.206	0.193	0.183
$\beta(\phi, \gamma)$ with t-stat in parentheses							
0.200	0.794	1.162	1.462	2.246	2.261	2.149	1.999
	(0.353)	(0.537)	(0.706)	(1.422)	(1.702)	(1.891)	(2.024)
0.500	0.257	0.355	0.434	0.611	0.598	0.557	0.511
	(0.396)	(0.573)	(0.733)	(1.388)	(1.631)	(1.790)	(1.900)
0.700	0.026	0.071	0.106	0.193	0.194	0.182	0.168
	(0.079)	(0.226)	(0.358)	(0.891)	(1.084)	(1.210)	(1.296)
0.900	−0.217	−0.208	−0.196	−0.118	−0.087	−0.067	−0.053
	(−1.437)	(−1.452)	(−1.449)	(−1.303)	(−1.204)	(−1.129)	(−1.072)
0.950	−0.357	−0.355	−0.341	−0.209	−0.153	−0.119	−0.096
	(−3.211)	(−3.425)	(−3.554)	(−3.528)	(−3.373)	(−3.246)	(−3.149)
0.980	−0.389	−0.382	−0.359	−0.202	−0.146	−0.113	−0.091
	(−5.096)	(−5.507)	(−5.731)	(−5.568)	(−5.288)	(−5.078)	(−4.925)
Panel B: Predictability of **Nominal** Interest Rate by Money-Adjusted Habit $R^2(\phi, \gamma)$							
0.200	0.059	0.078	0.098	0.201	0.245	0.275	0.296
0.500	0.048	0.069	0.091	0.203	0.252	0.285	0.308
0.700	0.030	0.050	0.072	0.193	0.246	0.283	0.309
0.900	0.006	0.024	0.052	0.229	0.303	0.350	0.381
0.950	0.004	0.029	0.068	0.294	0.373	0.419	0.448
0.980	0.006	0.039	0.085	0.313	0.382	0.421	0.445
$b(\phi, \gamma)$ with t-stat in parentheses							
0.200	−5.495	−6.114	−6.576	−7.201	−6.723	−6.113	−5.525
	(−2.595)	(−3.031)	(−3.434)	(−5.206)	(−5.921)	(−6.406)	(−6.744)
0.500	−1.434	−1.645	−1.802	−2.017	−1.877	−1.700	−1.532
	(−2.336)	(−2.828)	(−3.280)	(−5.242)	(−6.025)	(−6.558)	(−6.931)
0.700	−0.574	−0.706	−0.806	−0.964	−0.900	−0.815	−0.733
	(−1.834)	(−2.390)	(−2.898)	(−5.080)	(−5.943)	(−6.531)	(−6.945)
0.900	−0.114	−0.228	−0.314	−0.442	−0.402	−0.354	−0.311
	(−0.775)	(−1.646)	(−2.434)	(−5.662)	(−6.845)	(−7.618)	(−8.148)
0.950	−0.072	−0.187	−0.267	−0.342	−0.295	−0.251	−0.216
	(−0.642)	(−1.786)	(−2.805)	(−6.713)	(−8.017)	(−8.829)	(−9.369)
0.980	−0.067	−0.156	−0.211	−0.232	−0.194	−0.162	−0.138
	(−0.817)	(−2.082)	(−3.174)	(−7.008)	(−8.164)	(−8.858)	(−9.311)

aggregate is included, however, for parameter values above $\gamma = 0.40$, the R^2 exceeds 30%. This evidence is very useful since it suggests that a real term structure model cannot explain the dynamics of the nominal yield curve on its own. Moreover, the persistent effect generated by the habit stock helps explain the relationship between interest rates and lagged consumption.

IX. The Expectations Hypothesis

The expectations hypothesis of interest rates, hereafter EH, is one of the most frequently debated and studied financial relationships. If the EH were correct, at least in a statistical sense, one could use implied forward rates to obtain a simple unbiased proxy for the expected future spot rate. However, most of the empirical evidence rejects the EH and suggests the existence of a time-varying risk premium.[31] The extent and importance of the deviations are such that the empirical literature has used the size of the bias with respect to the EH predictions as a separate moment condition in building model specification tests. Such a metric is directly related to the properties of the conditional second moments of interest rates.[32]

We explore the extent to which monetary and habit factors can explain the time variation of the forward risk premium. Let the forward interest rate at time t for an instantaneous forward contract beginning at time $T = t + \tau$ be $f(t, T)$. The instantaneous forward rate is $-\frac{\partial}{\partial T} ln\, N(t, \tau)$. Taking the derivative of the log-price of the bond, we obtain

$$
\begin{aligned}
f(t,\tau) &= -\frac{\partial}{\partial \tau} lnN(t,\tau) \\
&= \rho - \sum_{i=1}^{2} \frac{\frac{\partial}{\partial \tau}\Lambda_{Yi}(\tau)Y_t + \frac{\partial}{\partial \tau}\Lambda_{\ell_i}(\tau)\ell_{it} + \frac{\partial}{\partial \tau}\Lambda_{\ell_i Y}(\tau)\ell_{it}Y_t + \frac{\partial}{\partial \tau}\Lambda_{0i}(\tau)}{(\Lambda_{Yi}(\tau)Y_t + \Lambda_{\ell_i}(\tau)\ell_{it} + \Lambda_{\ell_i Y}(\tau)\ell_{it}Y_t + \Lambda_{0i}(\tau))}.
\end{aligned}
$$

The nominal rate is given by

$$
R_t = \frac{Y_t(\ell_{1t} + \ell_{2t})}{\left[\sum_{i=1}^{2}\left(\Gamma_{Yi}Y_t + \Gamma_{\ell_i}\ell_{it} + \Gamma_{\ell_i Y}\ell_{it}Y_t + \Gamma_{0i}\right)\right]}.
$$

The EH postulates that the difference between the forward rate $f(t, \tau)$ and the expected future spot interest rate $E_t(R_{t+\tau})$ is constant, that is,

[31] Bekaert, Hodrick, and Marshall (2001) suggest a different explanation and investigate whether the violation of the EH in the U.S. data may be the result of a peso problem, whereby a high interest rate regime occurred less frequently in the U.S. sample than was rationally anticipated.

[32] The Campbell and Shiller (1991) tests of the EH focus on the slope coefficient properties of a regression of future yield changes on the current slope of the term structure. Since such a slope coefficient is a ratio between a conditional covariance and a conditional variance, the ability of a model to reproduce the empirical violations of the EH is a function of the ability of the model to reproduce the empirical conditional second moments of the interest rates.

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha.$$

However, in our model the forward risk premium depends on the levels of the nominal risk factors ℓ_{it} and the surplus-consumption ratio S_t. Thus, the model may provide a possible explanation for the EH violation. The nonlinear dependence of the nominal rate on the model factors does not permit analytical solutions for $E_t(R_{t+\tau})$. However, we can compute $E_t(R_{t+\tau})$ numerically using standard methods. To assess the forward premium's time variation and dependence on the monetary and habit factors, we regress the forward premium on the monetary factors ℓ_{it} and the inverse consumption-surplus ratio Y_t

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha + \beta_{1i}\ell_{it} + \beta_2 Y_t + \varepsilon_t.$$

We then assess (a) whether the forward premium is constant by testing $H_0 : \beta_{1i} = \beta_2 = 0$, and (b) the relative contribution of the nominal and habit factors to the forward risk premium's time variation. The results are given in Table VI. To gain insight into the reasons for the strong rejection of the EH, we decompose the total time variation of the forward premium into two components, the nominal factor and the habit factor. We find that at a 3-month horizon, 87% of the volatility of the forward premium is due to the nominal factors and 13% is due to the habit factor. At a 1-year horizon, the importance of the habit factor increases to 62%. Especially at short horizons, the rejection of the EH is mainly due to the time variation in the risk premium of nominal shocks.

Campbell and Shiller regressions. Campbell and Shiller (1991) regress the change in the constant time-of-maturity yield onto the current slope of the yield curve. To determine the time variation of the forward risk premium, we explore the following question: "If we generate term structure data using our structural model and run Campbell-Shiller (1991) type regressions, do we find the same pattern in the slope coefficients?" Let y_t^τ be the time-t yield on a Treasury bond with maturity $t+\tau$. Given a sampling frequency equal to m units of time, consider the regression

$$y_{t+m}^{n-m} - y_t^n = \alpha + \beta \underbrace{\left[\left(\frac{m}{n-m}\right)\left(y_t^n - y_t^m\right)\right]}_{S_t^{n,m}} + \varepsilon_t .$$

The EH suggests that $\beta = 1$. Campbell and Shiller (1991) test this hypothesis and find not only that the slope coefficient is different from one, but that it is negative. Thus, an increase in the slope of the term structure is followed by a decrease in long-term yields. At a 7-year maturity horizon, the empirical slope coefficient is about -3. These results have been proven to be robust. In fact, a large empirical literature now considers the slope coefficients of such regressions as moment conditions in building tests of model specification.

We generate data using the model and run a Campbell–Shiller type regression for each simulated run. We then test whether the model-implied moments are significantly different from the empirical ones. Since the slope coefficient is

Table VI
Test of Expectations Hypothesis

The table presents the results of a test for the unbiased expectations hypothesis. We linearize the forward premium and investigate the following specification:

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha + \beta_{1i} l_{ti} + \beta_2 S_t + \varepsilon_t.$$

Columns 1 and 2 present the results of the test of the unbiased expectation hypothesis, that is whether the forward premium is constant at different horizons (different values of τ) from 3 months to 5 years. Columns 3 and 4 quantify the relative contribution of the monetary and habit factors to the total variation in the forward premium.

			Relative Factor Contribution	
	J-stat	p-value	Monetary	Habit
3 months	6.14	0.00	87%	13%
6 months	8.06	0.00	73%	27%
1 year	9.60	0.00	62%	38%
2 years	16.57	0.00	64%	36%
3 years	33.29	0.00	61%	39%
5 years	65.51	0.00	57%	43%
Joint Test (All Maturities):			p-value = 0.00	

the ratio of the covariance between the left- and right-hand side variable over the variance of the right-hand side variable, we can design a GMM test using the slope coefficient as a moment condition. Let $\beta(\hat{\Theta})$ be the model-implied slope coefficient of the Campbell–Shiller regressions, given the set of estimated structural parameters $\hat{\Theta}$. Let $\hat{\beta}$ be the empirical slope coefficient obtained by rerunning the Campbell–Shiller regression on the data set. We simulate the model 1,000 times and test whether the simulated moments are close to those obtained from the data.

Table VII summarizes the results. Both the absolute levels of the slope coefficient and their patterns as a function of maturity closely mirror the Campbell and Shiller results. The slope coefficient at a 1-year horizon implied by the structural model is -0.02 compared to a value of -0.58 obtained empirically. As the horizon increases, the slope coefficients decrease as in Campbell and Shiller. At a 7-year horizon, the implied slope regression coefficient is -1.86 while the empirical Campbell–Shiller value is -2.13. We run chi-square tests of the null hypothesis that $H0 : \beta(\hat{\Theta}) = 1$ and that the two sets of coefficients are equal, that is, $H0 : \beta(\hat{\Theta}) = \hat{\beta}_{cs}$. We find that the implied values of the Campbell–Shiller regression coefficients strongly reject the EH at any confidence level. Additionally, the implied slope coefficients $\beta(\hat{\Theta})$ are not significantly different from those obtained by Campbell and Shiller, with p-values ranging between 0.07 and 0.44. The p-value of a joint test for all maturities is 0.14.

Why does the model succeed? Most traditional affine reduced-form models of the term structure assume that the market risk premium is proportional to the volatility of the latent factors. Duffee (2002), Duarte (2004), Dai and Singleton

Table VII

Campbell and Shiller Regressions

This table reports the Campbell and Shiller (1991) regressions. The main regression equation is

$$R_{t+m}^{n-m} - R_t^n = \alpha + \beta \left(\frac{m}{n-m} \right) (R_t^n - R_t^m) + \varepsilon_t,$$

where R_t^n is the yield of a bond with maturity n at time t. The expectations hypothesis implies that the coefficient β is equal to one. The value of m is taken to be 1 month. The first row shows the results of Campbell and Shiller regressions on a sample ranging between 1960 and 2000. The second row shows the values of the same β coefficient *implied* by the structural model at the estimated values of the structural parameters. Standard errors are given in parentheses.

	1y	2y	3y	5y	7y	10y
Empirical β	−0.579	−0.955	−1.238	−1.723	−2.135	−2.621
Model β	−0.020	−0.339	−0.652	−1.274	−1.865	−2.492
Standard Error	0.378	0.456	0.519	0.628	0.706	0.860
p-value for $H0 : \beta(\theta) = \beta_{CS}$	0.069	0.088	0.129	0.237	0.351	0.440
p-value for $H0 : \beta(\theta) = 1$	0.003	0.002	0.001	0.000	0.000	0.000
Joint Test (All Maturities) $H0 : \beta(\theta) = \beta_{CS}$				p-value = 0.14		
Joint Test (All Maturities) $H0 : \beta(\theta) = 1$				p-value = 0.002		

(2000), and Backus et al. (1999) show that this assumption is an important limiting feature. Our model differs in that the equilibrium price of risk is not directly proportional to the local volatility of the pricing factors. Moreover, it can change sign, allowing for more flexibility in the expected returns dynamics. The properties of the risk premium are such that the volatility of the returns can be high without necessarily implying a high expected bond return.

X. The Inflation Risk Premium

An additional important implication of the model regards the yield spread between nominal and index-linked bonds. Several papers in the empirical literature find that this spread exceeds the expected inflation rate, which indicates the existence of a large inflation risk premium. Using the U.K. data, Risa (2001) estimates that the average inflation risk premium has been above 100 basis points; Hördahl, Tristani, and Vestin (2005) find that in the euro-area it has fluctuated over time between 20 and 100 basis points. Ang and Bekaert (2005) obtain similar values using the U.S. Treasury data.[33]

This empirical evidence cannot be easily reconciled with standard asset pricing models. Lucas (2000), for instance, calibrates a cash-in-advance model and finds that in low inflation regimes the welfare costs of inflation are extremely small. Buraschi and Jiltsov (2005) show that in order for a structural model to

[33] An additional reason for the growing interest of the literature in the estimation of the inflation risk premium is that expected inflation rates are often computed by subtracting the inflation risk premium from the yield spread between nominal and indexed-linked bonds.

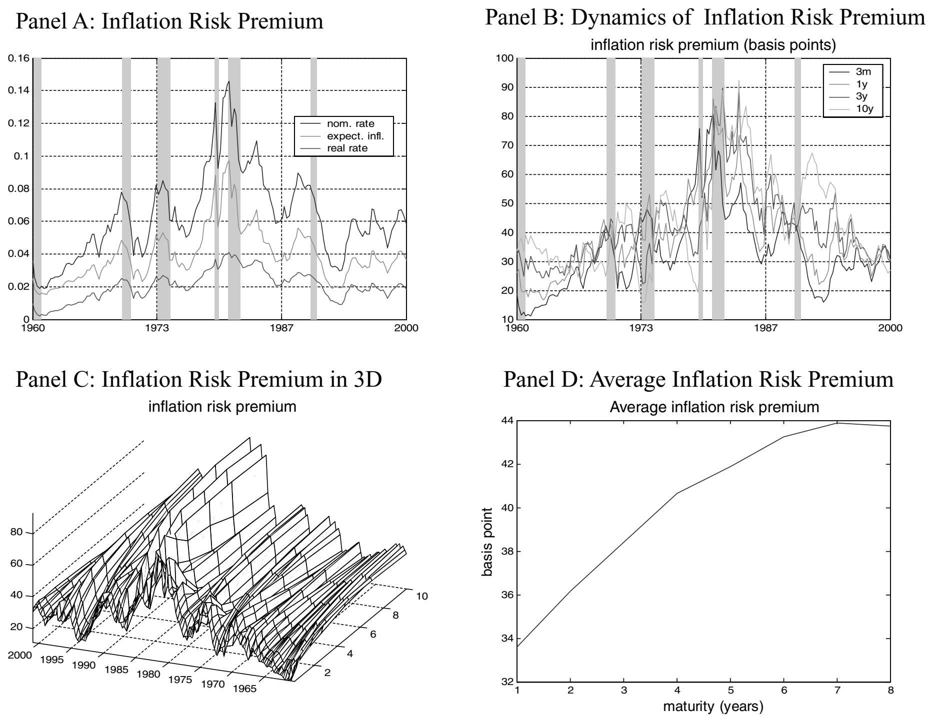

Figure 4. Inflation risk premium. Panel A plots the inflation risk premium over time with respect to nominal and realized real interest rates. Panel B shows the time variation of the inflation risk premium for different time horizons. The U.S. recession periods are marked as gray boxes. Panel C shows the three-dimensional evolution of the term structure of inflation risk premia. Panel D shows the average term structure of the inflation risk premia over the entire sample.

support realistic inflation risk premia, one needs to assume fiscal distortions and a large effective marginal tax rate.

In our model, the violation of the Fisher relationship between nominal and real interest rates is an equilibrium feature of the model and the inflation risk premium is positive even in the absence of fiscal distortions. The violation is generated by the fact that inflation reduces the real value of the monetary holdings that are used to finance the optimal consumption plan. With respect to traditional models with time-separable preferences, however, habit formation increases the volatility of the SDF and its covariance with inflation. Thus, habit formation can potentially generate inflation risk premia that are consistent with those observed empirically even abstracting from fiscal distortions. To explore this issue, in what follows we fix the estimated values of the structural parameters and compute the model-implied inflation risk premia.

The inflation risk premium is defined as $\mathrm{cov}_t[e^{-\rho\tau}\frac{u'(X_{t+\tau}-H_{t+\tau})}{u'_c(X_t-H_t)}; \frac{p^*_t}{p^*_{t+\tau}}]$, which is also equal to the difference between the value of a nominal bond, $N(t, t)$, and the value of a real bond adjusted by the expected change in the general price level, $B(t, \tau) \times E_t[\frac{P_t}{P_{t+\tau}}]$. When we use the estimated values of the structural parameters, we find that the average inflation rate risk premium is 44 basis points at an 8-year horizon, ranging over time between 20 and 90 basis points. The average term structure of this premium, calculated over the entire sample, is presented in Panel D of Figure 4. The term structure is upward sloping. In the short run, inflation is influenced by the short-term history of monetary policy. In the longer run, greater inflation uncertainty and longer bond duration translate into a higher premium on nominal bonds. Moreover, duration amplifies the price impact of inflation on long-term bonds so that long nominal bonds require higher risk premia.

Panel B of Figure 4 illustrates the dynamics of the inflation risk premia for different investment horizons. During periods of high nominal interest rates and inflation, such as during the 1982 recession, the inflation risk premium increased. Panel C of Figure 4 depicts the three-dimensional evolution of the inflation risk premium in both time and maturity domains and suggests that a structural model with habit persistence can support large, time-varying inflation risk premia that are consistent with those found by the empirical literature.[34]

XI. Conclusion

In the last 10 years, models with time-nonseparable preferences have been the focus of several studies. Important examples of these models explore preferences with habit formation. Little is known, however, of these models' implications for the term structure of interest rates. This paper is the first to investigate a monetary model with (external) habit formation and derive closed-form solutions for the dynamics of both nominal and real yield curves.

The distinctive features of the model with respect to traditional specifications are three-fold. First, the price of risk is not a constant multiple of interest rate volatility and it is state dependent. In bad (good) states of the world, the implied curvature of the indirect utility function is higher (lower). This is useful to help explain the changes in the observed term premium over the business cycle. Second, interest rates are correlated with both current and lagged monetary and consumption innovations. Third, the drift of the short-term interest rate is nonlinear.

We document new empirical evidence of the extent to which habit persistence can help explain the dynamics of the term structure of interest rates. In particular, we find the following.

First, tests of the overidentifying yield curve pricing restrictions do not reject a model with habit formation. We find that, at the structural model's estimated parameter values, the model can simultaneously reproduce both the persistence of the conditional second moments of changes in bond yields and their

[34] It would be of great interest to extend the data set to also include the U.S. index-linked bonds. Unfortunately, the first issue of these securities occurred in 1997, the lack of available data thus far limits the reliability of a joint estimation of the model.

conditional first moments. We run an asymptotic GMM test based on the second moments of yield changes and find that the null hypothesis that the model is correctly specified is not rejected at any horizon between 3 months and 10 years.

Second, habit formation helps reproduce both the sign and magnitude of the interest rate deviations from the expectations hypothesis described by Campbell and Shiller (1991). The model-implied linear projection coefficients are negative and increasing in absolute value with the regression horizon.

Third, we investigate whether, at the estimated parameter values, habit persistence helps explain the lead-lag correlation between interest rates and money highlighted in the macroeconomics literature (King and Watson (1996)). We find that a predictive regression of future nominal interest rates on the model-implied nominal habit stock produces an R^2 in excess of 30%.

Fourth, since the model can account for deviations from the Fisher hypothesis, we investigate the spread between nominal and real interest rates and estimate the inflation risk premium. We find that the inflation risk premium accounts for about one-fourth of the nominal versus real interest rate spread. This premium is upward sloping and time varying. The average inflation risk premium is 44 basis points at an 8-year horizon and it ranges between 20 and 90 basis points. We find that this time variation plays a key role in explaining the rejection of the expectations hypothesis.

Appendix: Proofs

Proof of Proposition 1: A. Strong form solution of the diffusion processes. Consider $dY_t = k(\theta - Y_t)\,dt - \sigma_y(Y_t - \lambda)\,dW_t$. Let $Z_t = Y_t - \lambda$, so that $dZ_t = k(\theta_z - Z_t)\,dt - \sigma_y Z_t\,dW_t$ and $\theta_z = \theta - \lambda$, and make a change of variable using the function $\omega(t) = \exp\{[k + \frac{1}{2}\sigma_y^2]t + \sigma_y(W(t) - W(0))\}$. Using Ito's rule and noting that $d\omega_t = \omega_t(k + \sigma_y^2)\,dt + \omega_t\sigma_y\,dW_t$, we obtain

$$
\begin{aligned}
d(\omega_t Z_t) &= \omega_t\,dZ_t + Z_t\,d\omega_t + \langle d\omega_t\,dZ_t\rangle \\
&= k\theta_z\omega_t\,dt,
\end{aligned}
$$

which can be directly integrated to yield

$$
Z_t = \frac{1}{\omega_t}\left[Z_0\omega_0 + k\theta_z\int_0^t \omega_s\,ds\right].
$$

Transforming back to $Y(t)$, we have

$$
Y(t) = \lambda + \frac{1}{\omega_t}\left[(Y_0 - \lambda) + k(\theta - \lambda)\int_0^t \omega_s\,ds\right]
$$

$$
\omega_s = \exp\left\{\left(k + \frac{1}{2}\sigma_y^2\right)s + \sigma_y(W_s - W_0)\right\}.
$$

B. Existence of a stationary density. Sufficient conditions for the existence of a stationary density are (1) $\sigma^2(y) > 0$ in the interior of the process's support (λ, ∞) and (2) both boundaries are entrance boundaries, that is, $\int_\lambda^\infty m(y)\,dy < \infty$ and $\int_\lambda^x s(y)\,dy = \int_x^\infty s(y)\,dy = \infty, \forall x \in (\lambda, \infty)$ (theorems 5.7 and 5.13 of Karatzas and Shreve (1991), p. 335; for applications Conley et al. (1997) and Ait-Sahalia (1996)), where $s(y)$ and $m(y)$ are the scale function and speed density of the process

$$s(y) = \exp\left[-\int^y \frac{2\mu(v)}{\sigma^2(v)}\,dv\right],\ L < y < U$$

$$m(y) = \frac{1}{\sigma^2(y)s(y)}.$$

Substituting the drift and volatility of the dY_t process, $s(y) = \exp[-\int^y \frac{2k(\theta - v)}{\sigma_y^2(v-\lambda)^2}\,dv]$. Integrating by parts, we have

$$s(y) = (y - \lambda)^{\frac{2k}{\sigma_y^2}} \exp\left[\frac{2k(\theta - y)}{\sigma_y^2(y - \lambda)}\right].$$

For the lower boundary, as $y \to \lambda$, if $\frac{2k\theta}{\sigma_y^2} > 0$ the scale function $s(y)$ is dominated by $\exp[\frac{2k(\theta - y)}{\sigma_y^2(y-\lambda)}]$ and $\int_0^x (y-\lambda)^{\frac{2k}{\sigma_y^2}} \exp[\frac{2k(\theta-y)}{\sigma_y^2(y-\lambda)}] = \infty$. For the upper boundary, as $y \to \infty$, if $\frac{2k\theta}{\sigma_y^2} > 0$ $s(y)$ is dominated by $(y-\lambda)^{\frac{2k}{\sigma_y^2}}$, whose integral is $\frac{1}{\frac{2k}{\sigma_y^2}+1}(y-\lambda)^{\frac{2k}{\sigma_y^2}+1}$. Thus, if $2k + \sigma_y^2 > 0$, we have $\int_x^\infty (y-\lambda)^{\frac{2k}{\sigma^2}} \exp[\frac{2k(\theta-y)}{\sigma_y^2(y-\lambda)}] = \infty$. The last condition requires that $\int_0^\infty m(v)\,dv < \infty$

$$\int_0^\infty m(v)\,dv = \int_0^\infty \frac{1}{\sigma_y^2(v-\lambda)^2}(v-\lambda)^{-\frac{2k}{\sigma^2}} \exp\left[-\frac{2k(\theta - v)}{\sigma_y^2(v-\lambda)}\right] dv$$

$$= \frac{1}{\sigma_y^2}\int_0^\infty \exp\left[-\frac{2k(\theta - v)}{\sigma_y^2(v-\lambda)}\right](v-\lambda)^{\frac{-2k-2\sigma_y^2}{\sigma_y^2}}\,dv.$$

For the upper boundary, as $v \to \infty$, the exponential converges to $\exp\frac{2k}{\sigma_y^2}$; the integral of $v^{\frac{-2k-2\sigma_y^2}{\sigma_y^2}}$ is $-\frac{\sigma_y^2}{2k+\sigma_y^2}v^{\frac{-2k-\sigma_y^2}{\sigma_y^2}}$, which converges to a finite value if $-2k - \sigma_y^2 < 0$ or $2k + \sigma_y^2 > 0$. For the lower boundary, as $v \to \lambda$, if $\frac{2k\theta}{\sigma_y^2} > 0$ the exponential goes to zero and it dominates the behavior of the integrand.

C. Functional form of the stationary density. Consider first the case $\lambda = 0$. If it exists, the stationary density $p(x)$ of the previous diffusion process is equal to the normalized speed function, that is, $p(x) = \mathcal{N}\cdot m(x)$, where $\mathcal{N}$ is the normalization constant. Given the previous solution for $m(x)$, the stationary distribution must take the form $p(x) = \mathcal{N} \cdot x^a \exp(\frac{b}{x})$. This is an Inverted Gamma density with $a = -2(1 + \frac{k}{\sigma^2})$ and $b = -2\frac{k\theta}{\sigma^2}$.

The solution can be easily verified by checking that $p(x)$ solves the Fokker–Planck equation $\frac{d}{dy}[\mu(y)p(y)] - \frac{1}{2}\frac{d^2}{dy^2}[\sigma^2(y)p(y)] = 0$, which can be rewritten (see Wong (1964)) as $[p(x)\sigma^2(x)]' = 2\mu(x)p(x)$. Substituting the guess $p(x) = \mathcal{N} \cdot x^a \exp(\frac{b}{x})$, we obtain

$$(\sigma^2 a + 2\sigma^2)\left[x^{a+1}\exp\left(\frac{b}{x}\right)\right] - b\sigma^2\left[x^a \exp\left(\frac{b}{x}\right)\right]$$
$$= 2k\theta\left[x^a \exp\left(\frac{b}{x}\right)\right] - 2k\left[x^{a+1}\exp\left(\frac{b}{x}\right)\right].$$

Matching the coefficients, we have $\sigma^2(2 + a) = -2k$ and $-b\sigma^2 = 2k\theta$. This implies that $a = -2(1 + \frac{k}{\sigma^2})$ and $b = -2\frac{k\theta}{\sigma^2}$. Notice that the Inverse Gamma distribution is well defined if and only if $b < 0$ and $a < -1$. These are satisfied if conditions (*C1*) and (*C2*), respectively, are satisfied: $k\theta > 0$ and $2k + \sigma^2 > 0$.

For $\lambda > 0$, the diffusion can be easily obtained from the previous one using a change of variables, namely, $x = y - \lambda$. The result is $p(y) = \mathcal{N} \cdot (y - \lambda)^a \exp(\frac{b}{y-\lambda})$. This is an Inverted Gamma density with $a = -2(1 + \frac{k}{\sigma^2})$ and $b = -2\frac{k(\theta-\lambda)}{\sigma^2}$.

D. Second moments. Let us consider the canonical representation of Y_T^2. Taking the expected value and differentiating with respect to T, we obtain

$$\frac{dE_0(Y_T^2)}{dT} = \nu_0(T) + \nu_1 E_0(Y_T^2),$$

with $\nu_1 = (\sigma^2 - 2k)$ and $\nu_0(T) = 2(k\theta - \lambda\sigma^2)E_t(Y_T) + \lambda^2\sigma^2$. The previous differential equation admits the following solution:

$$E_0 Y^2(T) = e^{(\sigma^2-2k)T}Y_0^2 + \int_0^T \nu_0(s)e^{(\sigma^2-2k)(T-s)}\,ds.$$

If condition (C2) is satisfied, the result follows. Q.E.D.

Proof of Lemma 1. (Conditional Moments of Product): Consider a linear system of two mean-reverting Ito processes, ξ_{1t} and ξ_{2t}

$$d\xi_{1t} = k_{\xi_1}(\theta_{\xi_1} - \xi_{1t})\,dt + (\xi_{1t} - \lambda_{\xi_1})\left[\upsilon dW_t^2 + \sigma_{\xi_1} dW_t^1\right]$$

$$d\xi_{2t} = k_{\xi_2}(\theta_{\xi_2} - \xi_{2t})\,dt + (\xi_{2t} - \lambda_{\xi_2})\sigma_{\xi_2} d\,W_t^2, \; E\left(dW_t^1 \cdot dW_t^2\right) = \rho dt.$$

We will prove that the conditional expectation of their product, $q_t = \xi_{1t}\xi_{2t}$, is equal to

$$E_t[q_{t+\tau}] = A_3(\tau; \Theta_{\xi_1}, \Theta_{\xi_2})q_t + A_1(\tau; \Theta_{\xi_1}, \Theta_{\xi_2})\xi_{1t}$$
$$+ A_2(\tau; \Theta_{\xi_1}, \Theta_{\xi_2})\xi_{2t} + A_0(\tau; \Theta_{\xi_1}, \Theta_{\xi_2}). \tag{A1}$$

The diffusion of $q_t = \xi_{1t}\xi_{2t}$ is

$$
\begin{aligned}
dq_t = {} & \xi_{2t}\big[k_{\xi_1}(\theta_{\xi_1} - \xi_{1t})\,dt + (\xi_{1t} - \lambda_{\xi_1})\big[\upsilon\, dW_t^2 + \sigma_{\xi_1}\, dW_t^1\big]\big] \\
& + \xi_{1t}\big(k_{\xi_2}(\theta_{\xi_2} - \xi_{2t})\,dt + \sigma_{\xi_2}(\xi_{2t} - \lambda_{\xi_2})\,dW_t^2\big) \\
& + \rho\sigma_{\xi_2}(\xi_{2t} - \lambda_{\xi_2})\sigma_{\xi_1}(\xi_{1t} - \lambda_{\xi_1})\,dt + (\xi_{2t} - \lambda_{\xi_2})\sigma_{\xi_2}(\xi_{1t} - \lambda_{\xi_1})\upsilon\, dt.
\end{aligned}
$$

The stochastic process q_t follows

$$
dq_t = [a_3 + h_{\xi_2}\xi_{2t} + h_{\xi_1}\xi_{1t} + h_q q_t]\,dt + \Sigma_q\, dW_t,
$$

with

$$
\begin{cases}
h_{\xi_2} = k_{\xi_1}\theta_{\xi_1} - \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_1} - \upsilon\sigma_{\xi_2}\lambda_{\xi_1} & a_1 = k_{\xi_2}\theta_{\xi_2} \\
h_{\xi_1} = k_{\xi_2}\theta_{\xi_2} - \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_2} - \upsilon\sigma_{\xi_2}\lambda_{\xi_1} & a_2 = k_{\xi_1}\theta_{\xi_1} \\
h_q = -k_{\xi_1} - k_{\xi_2} + \rho\sigma_{\xi_2}\sigma_{\xi_1} + \upsilon\sigma_{\xi_2} & a_3 = \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_2}\lambda_{\xi_2} + \upsilon\sigma_{\xi_1}\lambda_{\xi_1}\lambda_{\xi_2}.
\end{cases}
\tag{A2}
$$

Consider the following three-dimensional process $f_t = [\xi_{2t}, \xi_{1t}, q_t]$. We can describe the dynamics of the process as

$$
df_t = (A_0 + A_1 f_t)\,dt + \Sigma_f\, dW,
$$

where

$$
A_0 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -k_{\xi_2} & 0 & 0 \\ 0 & -k_{\xi_1} & 0 \\ h_{\xi_2} & h_{\xi_1} & h_q \end{bmatrix}.
$$

The system is linear, and the expected value $E_t\,[f_{t+\tau}]$ can be calculated as

$$
E_t[f_{t+\tau}] = \Psi(t+\tau)f_t + \int_t^{t+\tau} \Psi((t+\tau) - s)A_0\, ds,
$$

with $\Psi(\tau) = U\exp(\Lambda \cdot \tau)U^{-1}$ where Λ is a diagonal matrix of eigenvalues of A_1 and U is the matrix of associated eigenvectors. We can find that

$$
\Lambda = \begin{pmatrix} h_q & & \\ & -k_{\xi_1} & \\ & & -k_{\xi_2} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & -\dfrac{h_q + k_{\xi_2}}{h_{\xi_2}} \\ 0 & -\dfrac{h_q + k_{\xi_1}}{h_{\xi_1}} & 0 \\ 1 & 1 & 1 \end{pmatrix}.
$$

Simple matrix multiplication gives

$$\Psi(\tau) = U\exp(\Lambda\cdot\tau)U^{-1} = \begin{pmatrix} e^{-k_{\xi_2}\tau} & 0 & 0 \\ 0 & e^{-k_{\xi_1}\tau} & 0 \\ \dfrac{(e^{h_q\tau} - e^{-k_{\xi_2}\tau})h_{\xi_2}}{h_q + k_{\xi_2}} & \dfrac{(e^{h_q\tau} - e^{-k_{\xi_1}\tau})h_{\xi_1}}{h_q + k_{\xi_1}} & e^{h_q\tau} \end{pmatrix}.$$

Let us define the vector $e_3 = (0, 0, 1)$. Then, the expected value of q_t is

$$E_t[q_{t+\tau}] = E_t[e_3 f_{t+\tau}] = e_3\Psi(\tau)f_t + \int_t^{t+\tau} e_3\Psi(t+\tau-s)A_0\,ds.$$

After some algebra we obtain

$$\begin{aligned} E_t[q_{t+\tau}] &= q_t e^{h_q\tau} + \xi_{1t}\frac{(e^{h_q\tau} - e^{-k_{\xi_1}\tau})h_{\xi_1}}{h_q + k_{\xi_1}} + \xi_{2t}\frac{(e^{h_q\tau} - e^{-k_{\xi_2}\tau})h_{\xi_2}}{h_q + k_{\xi_2}} \\ &\quad + a_3\frac{1-e^{h_q\tau}}{h_q} + a_2\frac{h_{\xi_1}}{h_q + k_{\xi_1}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_1}\tau}}{k_{\xi_1}}\right] \\ &\quad + a_1\frac{h_{\xi_2}}{h_q + k_{\xi_2}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_2}\tau}}{k_{\xi_2}}\right]. \end{aligned} \tag{A3}$$

Substituting back the values for $a_1, a_2, a_3, h_{\xi_2}, h_{\xi_1}$, and h_q using the equations (*A2*), we obtain the solution in terms of the original parameters. That is,

$$\begin{aligned} E_t[q_{t+\tau}] &= A_3(\tau;\Theta_{\xi_1},\Theta_{\xi_2})q_t + A_1(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{1t} \\ &\quad + A_2(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{2t} + A_0(\tau;\Theta_{\xi_1},\Theta_{\xi_2}), \end{aligned} \tag{A4}$$

where

$$\begin{cases} A_3(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = e^{h_q\tau},\ A_1(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = \dfrac{(e^{h_q\tau} - e^{-k_{\xi_1}\tau})h_{\xi_1}}{h_q + k_{\xi_1}}, \\ A_2(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = \dfrac{(e^{h_q\tau} - e^{-k_{\xi_2}\tau})h_{\xi_2}}{h_q + k_{\xi_2}} \\ A_0(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = a_3\dfrac{1-e^{h_q\tau}}{h_q} + a_2\dfrac{h_{\xi_1}}{h_q + k_{\xi_1}}\left[\dfrac{1-e^{h_q\tau}}{-h_q} - \dfrac{1-e^{-k_{\xi_1}\tau}}{k_{\xi_1}}\right] \\ \qquad\qquad + a_1\dfrac{h_{\xi_2}}{h_q + k_{\xi_2}}\left[\dfrac{1-e^{h_q\tau}}{-h_q} - \dfrac{1-e^{-k_{\xi_2}\tau}}{k_{\xi_2}}\right], \end{cases} \tag{A5}$$

with Θ_{ξ_i} being the structural parameters of the diffusion processes $\Theta_{\xi_i} \equiv$

$[k_{\xi_i}, \theta_{\xi_i}, \sigma_{\xi_i}, \lambda]$. Q.E.D.

PROPOSITION 2. (General Price Level): The general equilibrium price level is obtained from the equilibrium rate of substitution between the money stock m_t and consumption.[35] We have

$$\frac{1}{P_t} = E_t\left[\int_t^{\infty} e^{-\rho(s-t)} \frac{u_m(C_s, m_s, H_s)}{u_c(C_t, m_t, H_t)} \frac{1}{P_s}\, ds\right].$$

In the case of the log-utility function $u(c, m, H) = \log(Cm^{\gamma} - H)$, we obtain

$$\begin{aligned} \frac{1}{P_t} &= \gamma \frac{C_t}{Y_t} \int_t^{\infty} e^{-\rho(s-t)} E_t\left[Y_s \frac{1}{M_s}\right] ds \\ &= \gamma \frac{C_t}{Y_t} \frac{1}{L_t M_t} \int_t^{\infty} e^{-(\rho+\mu_M)(s-t)} E_t[Y_s L_s]\, ds. \end{aligned} \tag{A6}$$

To solve for the price level we need to solve for the expectation under the integral. For simplicity, let us first consider the univariate case. Let $q_t = Y_t L_t$ and $f_t = [L_t, Y_t, q_t]$. Using Ito's rule, it is easy to show that

$$df(t) = [A_0 + A_1 f(t)]\, dt + \Sigma\, dW_t,$$

with

$$A_0 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -k_\ell & 0 & 0 \\ 0 & -k_y & 0 \\ h_\ell & h_Y & h_q \end{bmatrix}$$

$$\begin{aligned} a_1 &= k_\ell \theta_\ell, \quad a_2 = k_y \theta_y, \quad a_3 = 0 \\ h_Y &= k_\ell \theta_\ell \\ h_\ell &= k_y \theta_y + (\lambda \sigma_{yc} \sigma_\ell \rho + \lambda \sigma_{y\ell} \sigma_\ell) \\ h_q &= -k_y - k_\ell - (\sigma_{yc} \sigma_\ell \rho + \sigma_{y\ell} \sigma_\ell). \end{aligned}$$

Let $\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1}$, where Λ is the diagonal matrix of the eigenvalues of A_1 and U is the associated eigenvector matrix. From Lemma 2, we have

$$E_t[f_{t+\tau}] = \Psi(t+\tau) f_t + \int_t^{t+\tau} \Psi((t+\tau) - s) A_0\, ds.$$

[35] See Bakshi and Chen (1996a) for an analytical derivation of this continuous-time first-order condition obtained as the continuous-time limit of a discrete-time economy.

Notice that

$$\Lambda = \begin{pmatrix} h_q & & \\ & -k_Y & \\ & & -k_\ell \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & -\dfrac{h_q + k_\ell}{h_2} \\ 0 & -\dfrac{h_q + k_y}{h_1} & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

with

$$\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1} = \begin{pmatrix} e^{-k_\ell \tau} & 0 & 0 \\ 0 & e^{-k_y \tau} & 0 \\ \dfrac{(e^{h_q \tau} - e^{-k_\ell \tau}) h_\ell}{h_q + k_\ell} & \dfrac{(e^{h_q \tau} - e^{-k_y \tau}) h_Y}{h_q + k_y} & e^{h_q \tau} \end{pmatrix}.$$

Therefore, using the result (*A4*), we have

$$E_t[Y_s L_s] = \sum_{i=1}^{2} (A_q(\tau) q_t + A_Y(\tau) Y_t + A_\ell(\tau) \ell_{it} + A_0(\tau)),$$

where

$$\begin{cases} A_q(\tau) = e^{h_q \tau}, A_Y(\tau) = \dfrac{\left(e^{h_q \tau} - e^{-k_y \tau}\right) h_Y}{h_q + k_y}, \quad A_\ell(\tau) = \dfrac{\left(e^{h_q \tau} - e^{-k_\ell \tau}\right) h_\ell}{h_q + k_\ell} \\ A_0(\tau) = a_3 \dfrac{1 - e^{h_q \tau}}{h_q} + a_2 \dfrac{h_Y}{h_q + k_y} \left[\dfrac{1 - e^{h_q \tau}}{-h_q} - \dfrac{1 - e^{-k_y \tau}}{k_y} \right] \\ \qquad + a_1 \dfrac{h_\ell}{h_q + k_\ell} \left[\dfrac{1 - e^{h_q \tau}}{-h_q} - \dfrac{1 - e^{-k_\ell \tau}}{k_\ell} \right]. \end{cases} \tag{A7}$$

Hence, the inverse price level is

$$\frac{1}{P_t} = \gamma \frac{C_t}{Y_t} \frac{1}{L_t M_t} \int_0^\infty e^{-(\rho + \mu_M)\tau} \left[\sum_{i=1}^{2} \left(A_q(\tau) q_t + A_Y(\tau) Y_t + A_\ell(\tau) \ell_{it} + A_0(\tau) \right) \right] d\tau.$$

In order for the integral to converge, the parameter $A_i(\tau)$ needs to be bounded. In addition to conditions (*C1*) and (*C2*), this requires additional constraints on the size of the covariance terms between the liquidity shocks and the dY process:

$$h_q = -k_y - k_{\ell_i} - (\sigma_{yc} \sigma_{\ell_i} \rho_{\ell_i c} + \sigma_{y \ell_i} \sigma_{\ell_i}) < 0.$$

Note that the expressions for the As are of the form $e^{\zeta s}$ or $\frac{1 - e^{\zeta t}}{\zeta}$. Thus, for convenience let us calculate the following integrals for a generic value ζ; we will later substitute their values as a function of the structural parameters:

$$\int_0^\infty e^{-(\rho+\mu_M)\tau} e^{\zeta\tau} d\tau = \int_0^\infty e^{-(\rho+\mu_M-\zeta)\tau} d\tau = \frac{1}{\rho+\mu_M-\zeta}$$

$$\int_0^\infty e^{-(\rho+\mu_M)\tau}\left[\frac{1-e^{\zeta\tau}}{\zeta}\right] d\tau = \frac{1}{\zeta}\left[\frac{1}{\rho+\mu_M} - \frac{1}{\rho+\mu_M-\zeta}\right]$$
$$= -\frac{1}{(\rho+\mu_M)(\rho+\mu_M-\zeta)}.$$

Using this result, consider the first term inside the integral, that is,

$$\Gamma_q \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_q(\tau)\, d\tau = \int_0^\infty e^{-(\rho+\mu_M)\tau} e^{h_q\tau} d\tau$$
$$= \frac{1}{\rho+\mu_M-h_q}.$$

Similarly,

$$\Gamma_\ell \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_\ell(\tau) d\tau = \frac{h_\ell}{h_q+k_\ell}\left[\frac{1}{\rho+\mu_M-h_q} - \frac{1}{\rho+\mu_M+k_\ell}\right]$$

$$\Gamma_Y \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_Y(\tau)\, d\tau = \frac{h_Y}{h_q+k_y}\left[\frac{1}{\rho+\mu_M-h_q} - \frac{1}{\rho+\mu_M+k_y}\right]$$

$$\Gamma_0 \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_0(\tau)\, d\tau = a_3\left[-\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)}\right]$$
$$+a_2\frac{h_Y}{h_q+k_y}\left[\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)} - \frac{1}{(\rho+\mu_M)(\rho+\mu_M+k_y)}\right]$$
$$+a_1\frac{h_\ell}{h_q+k_\ell}\left[\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)} - \frac{1}{(\rho+\mu_M)(\rho+\mu_M+k_\ell)}\right].$$

Summarizing,

$$\frac{1}{P_t} = \gamma\frac{C_t}{Y_t}\frac{1}{L_t M_t}\left[\Gamma_q(\tau)q_t + \Gamma_Y(\tau)Y_t + \Gamma_\ell(\tau)\ell_t + \Gamma_0(\tau)\right].$$

In the multivariate case, $L_t = \sum_i \ell_{it}$, the result is easily generalizable to

$$\frac{1}{P_t} = \gamma\frac{C_t}{Y_t}\frac{1}{L_t M_t}\left[\sum_{i=1}^{2}\left(\Gamma_{q_i}(\tau)q_{it} + \Gamma_{Y_i}(\tau)Y_t + \Gamma_{\ell_i}(\tau)\ell_{it} + \Gamma_{0i}(\tau)\right)\right],$$

where the additional index i refers to the parameters of the process ℓ_{it}.

The integral of the future values of the inverse surplus-consumption ratio $E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} Y_s ds$ and monetary factors $E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} \ell_{is} ds$ can be obtained using similar methods. Under the assumption that the two integrals converge, which requires $\rho + \mu_M > 0$, we apply Fubini's theorem to invert the order of integration. Moreover, the linearity of the drift of dY_t implies $E_t(Y_s) = \theta_y + (Y_t - \theta_y)e^{-k_y(s-t)}$, so that

$$E_t \int_t^{\infty} e^{-(\rho+\mu_M)(s-t)} Y_s\, ds = \frac{\theta_y}{\rho + \mu_M} + \frac{(Y_t - \theta_y)}{\rho + \mu_M + k_y}$$

$$E_t \int_t^{\infty} e^{-(\rho+\mu_M)(s-t)} \ell_{is} ds = \frac{\theta_\ell^i}{\rho + \mu_M} + \frac{\left(\ell_{is} - \theta_\ell^i\right)}{\rho + \mu_M + k_\ell^i}.$$

Proof of Proposition 3. (High-Order Conditional Moments): Consider $d\ell(t) = k(\theta - \ell)dt + \sigma(\ell - \ell_0)dW_t$. From Ito's rule,

$$d\left[\ell(t)^n\right] = n\ell(t)^{n-1} d\,\ell(t) + \frac{n(n-1)}{2}\ell(t)^{n-2}[\sigma_\ell(\ell_t - \ell_0)]^2\, dt.$$

Thus,

$$\begin{aligned} \frac{d}{dt}E_0\ell(t)^n &= E_0\ell(t)^n\left[-nk + \frac{n(n-1)}{2}\sigma_y^2\right] \\ &\quad + E_0\ell(t)^{n-1}\left[nk\theta - \lambda n(n-1)\sigma_y^2\right] \\ &\quad + E_0\ell(t)^{n-2}\left[\frac{n(n-1)}{2}\ell_0^2\sigma_y^2\right]. \end{aligned}$$

Let $V_0(t) \equiv E_0\ell(t)^n$. Then integrating between 0 and T, we have

$$V_0(T) - V_0(0) = \int_0^T \Psi_0(s)\, ds + \int_0^T \Psi_1 V_t(s)\, ds$$

$$\Psi_0(s) = E_0\ell(s)^{n-1}\left[nk\theta - \lambda n(n-1)\sigma_y^2\right] + E_0\ell(s)^{n-2}\left[\frac{n(n-1)}{2}\ell_0^2\sigma_y^2\right]$$

$$\Psi_1 = \left[-nk + \frac{n(n-1)}{2}\sigma_y^2\right].$$

Differentiating with respect to T we then have

$$V_0'(T) = \Psi_0(T) + \Psi_1 V(T),$$

which is known to have the solution $E_0\ell(T)^n = e^{\Psi_1 T}\ell(0)^n + \int_0^T \Psi_0(s)e^{\Psi_1(T-s)}\, ds$. Notice the dependence of $\Psi_0(T)$ on the conditional moments $E_0\ell(T)^{n-1}$ and $E_0\ell(T)^{n-2}$. The first conditional moment satisfies the following ODE

$$\frac{dE_t(\ell_T)}{dT} = k\theta - kE_t(\ell_T),$$

which has solutions $E_t(\ell_T) = \theta + (\ell_t - \theta)e^{-k(T-t)}$. All the other moments can be computed recursively.

The conditional variances of Y_T and ℓ_T and the central cross moment $E_t[(Y_T - E_t(Y_T))(\ell_T - E_t(\ell_T))]$ can be constructed using the same approach.

REFERENCES

Abel, Andrew B., 1990, Asset prices under habit formation and catching up with the Joneses, *American Economic Review* 80, 38–42.

Ait-Sahalia, Yacine, 1996, Nonparametric pricing of interest rate derivative securities, *Econometrica* 64, 527–560.

Ait-Sahalia, Yacine, 1999, Transition densities for interest rate and other nonlinear diffusions, *Journal of Finance* 54, 1361–1395.

Ait-Sahalia, Yacine, 2002, Maximum likelihood estimation of discretely sampled diffusions: A closed form approximation approach, *Econometrica* 70, 223–262.

Ang, Andrew, and Geert Bekaert, 2005, The term structure of real rates and expected inflation, Working paper, Columbia University.

Attanasio, Orazio P., and Guglielmo Weber, 1995, Is consumption growth consistent with intertemporal optimization? Evidence from the consumer expenditure survey, *Journal of Political Economy* 103, 1121–1157.

Backus, David, Chris I. Telmer, and Liuren Wu, 1999, Design and estimation of affine yield models, Working paper, GSIA, Carnegie Mellon University.

Bakshi, Gurdip S., and Zhiwu Chen, 1996a, Inflation, asset prices, and the term structure of interest rates in monetary economies, *Review of Financial Studies* 9, 241–275.

Bakshi, Gurdip S., and Zhiwu Chen, 1996b, The spirit of capitalism and stock prices, *American Economic Review* 86, 133–157.

Bekaert, Geert, 1996, The time variation of risk and return in foreign exchange markets: A general equilibrium perspective, *Review of Financial Studies* 9, 427–470.

Bekaert, Geert, Eric Engstrom, and Steven Grenadier, 2004, Stock and bond returns with moody investors, Working paper, Columbia University.

Bekaert, Geert, Robert Hodrick, and David Marshall, 2001, Peso problem explanations for term structure anomalies, *Journal of Monetary Economics* 48, 241–270.

Benninga, Simon, and Ariel Protopapadakis, 1983, Real and nominal interest rates under uncertainty: The Fisher theorem and the term structure, *Journal of Political Economy* 91, 856–867.

Boldrin, Michele, Lawrence J. Christiano, and Jonas D. Fisher, 2001, Habit persistence, asset returns, and the business cycle, *American Economic Review* 91, 149–166.

Boudoukh, Jacob, 1993, An equilibrium model of nominal bond prices with initiation-output correlation and stochastic volatility, *Journal of Money, Credit and Banking* 25, 636–665.

Buraschi, A., and Menini D., 2002, Liquidity risk and specialness, *Journal of Financial Economics* 64, 243–284.

Buraschi, Andrea, and Alexei Jiltsov, 2005, Initiation risk premia and the expectations hypothesis: A monetary model of the Treasury yield curve, *Journal of Financial Economics* 75, 429–490.

Campbell, John Y., and John H. Cochrane, 1998, By force of habit: A consumption-based explanation of aggregate stock market behavior, Working paper, http://www-gsb.uchicago.edu/fac/john.cochrane/research/Papers/habit.pdf.

Campbell, John Y., and John H. Cochrane, 1999, By force of habit: A consumption-based explanation of aggregate stock market behavior, *Journal of Political Economy* 107, 205–251.

Campbell, John Y., and Robert J. Shiller, 1991, Yield spreads and interest rates movements, a bird's eye view, *Review of Economic Studies* 58, 495–514.

Carroll, Christopher D., Jody Overland, and David N. Weil, 2000, Saving and growth in habit formation, *American Economic Review* 90, 341–355.

Chapman, David A., 1998, Habit formation and aggregate consumption, *Econometrica* 66, 1223–1230.

Chari, V. V., Lawrence J. Christiano, and Martin Eichenbaum, 1995, Inside money, outside money and short-term interest rates, NBER Working paper 5269.

Chen, Nai-Fu, Richard Roll, and Steven Ross, 1986, Economic forces and the stock market, *Journal of Business* 59, 383–403.

Chen, R., and L. Scott, 1993, Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates, *Journal of Fixed Income* 3, 14–31.

Cheridito, P., D. Filipovic, and R. L. Kimmel, 2007, Market price of risk specifications for affine models: Theory and evidence, *Journal of Financial Economics* (forthcoming).

Chetty, Ray, and Adam Szeidl, 2003, Consumption commitments and asset prices, Working paper, Harvard University.

Conley, Timothy G., Lars P. Hansen, Erzo G. J. Luttmer, and Jose A. Scheinkman, 1997, Short-term interest rates as subordinated diffusions, *Review of Financial Studies* 10, 525–577.

Constantinides, George, 1990, Habit formation: A resolution of the equity premium puzzle, *Journal of Political Economy* 98, 519–543.

Cox, John C., J.E. Ingersoll, and S.A. Ross, 1985, A theory of the term structure of interest rates, *Econometrica* 53, 385–407.

Dai, Qiang, 2002, Term structure dynamics in a model with stochastic internal habit, Working paper, New York University.

Dai, Qiang, and Kenneth Singleton, 2000, Specification analysis of affine term structure modes, *Journal of Finance* 50, 1943–1978.

Dai, Qiang, and Kenneth Singleton, 2002, Expectations puzzles, time-varying risk premia, and affine models of the term structure, *Journal of Financial Economics* 63, 415–441.

Daniel, Kent, and David Marshall, 1997, The equity premium puzzle and the risk-free rate puzzle at long horizons, *Macroeconomic Dynamics* 1, 452–484.

Detemple, Jerome B., and Fernando Zapatero, 1991, Asset prices in an exchange economy with habit formation, *Econometrica* 59, 1633–1657.

Duarte, Jefferson, 2004, Evaluating an alternative risk preference in affine term structure models, *Review of Financial Studies* 17, 379–404.

Duesenberry, James S., 1949, *Income, Saving, and the Theory of Consumer Behavior* (Harvard University Press, Cambridge, MA).

Duffee, Gregory, 2002, Term premia and interest rate forecasts in affine models, *Journal of Finance* 57, 405–443.

Duffie, Darrell, and Ken Singleton, 1997, An econometric model of the term structure of interest rate swap yields, *Journal of Finance* 52, 1287–1323.

Dunn, K., and K. Singleton, 1986, Modeling the term structure of interest rates under non-separable utility and durability of goods, *Journal of Financial Economics* 17, 27–55.

Dynan, Karen E., 2000, Habit formation in consumer preferences: Evidence from panel data, *American Economic Review* 90, 391–406.

Engle, Robert F., and Tim Bollerslev, 1986, Modelling the persistence of conditional variances, *Econometric Reviews* 5, 1–50.

Evans, Martin D., 1998, Real rates, expected inflation, and inflation risk premia, *Journal of Finance* 53, 187–218.

Fama, Eugene F., 1976, Inflation uncertainty and expected returns on Treasury bills, *Journal of Political Economy* 84, 427–448.

Fama, Eugene F., 1981, Stock returns, real activity, inflation, and money, *American Economic Review* 74, 545–565.

Fama, Eugene F., 1990, Term structure forecasts of interest rates, inflation, and real returns, *Journal of Monetary Economics* 25, 59–76.

Fama, Eugene F., and Robert R. Bliss, 1987, The information in long-maturity forward rates, *American Economic Review* 77, 680–692.

Fama, Eugene F., and M. R. Gibbons, 1982, Inflation, real returns and capital investment, *Journal of Monetary Economics* 9, 297–323.

Feenstra, R., 1986, Function equivalence between liquidity costs and utility of money, *Journal of Monetary Economics* 17, 271–291.

Feller, W., 1951, Two singular diffusion problems, *Annals of Mathematics* 54, 173–182.

Ferson, Wayne E., and George Constantinides, 1991, Habit persistence and durability in aggregate consumption: Empirical tests, *Journal of Financial Economics* 29, 199–240.

Fiorito, Riccardo, and Tryphon Kollintzas, 1994, Stylized facts of business cycles in the G7 from a real business cycles perspective, *European Economic Review* 38, 235–269.

Fuhrer, Jeffrey C., 2000, Habit formation in consumption and its implications for monetary-policy models, *American Economic Review* 90, 367–390.

Green, Richard C., and Bernt A. Odegaard, 1997, Are there tax effects in the relative pricing of U.S. government bonds? *Journal of Finance* 52, 609–633.

Hansen, Lars P., 1982, Large sample properties of generalized method of moments estimators, *Econometrica* 50, 1029–1054.

Hansen, Lars P., and Ravi Jagannathan, 1991, Implications of security market data for models of dynamic economies, *Journal of Political Economy* 99, 225–262.

Hansen, Lars P., and Jose A. Scheinkman, 1995, Back to the future: Generating moment implications for continuous-time Markov processes, *Econometrica* 63, 767–804.

Harvey, Campbell R., 1988, The real term structure and consumption growth, *Journal of Financial Economics* 22, 305–334.

Heaton, John C., 1995, An empirical investigation of asset pricing with temporally dependent preference specifications, *Econometrica* 63, 681–717.

Heien, Dale, and Cathy Duhram, 1991, A test of habit formation hypothesis using household data, *Review of Economics and Statistics* 74, 189–199.

Hördahl, Peter, Oreste Tristani, and David Vestin, 2005, The term structure of initiation risk premia and macroeconomic dynamics, Working paper, European Central Bank.

Karatzas, I., and S. E. Shreve, 1991, *Brownian Motion and Stochastic Calculus* (Springer-Verlag: New York, NY).

Karlin, S., and H. M. Taylor, 1981, *A Second Course in Stochastic Processes* (Academic Press: New York, NY).

King, R., and M. Watson, 1996, Money, prices and the business cycle, *Review of Economics and Statistics* 78, 35–53.

Kwon, Heon-Chul, 1992, The time variant term premium of interest rates Ph.D. Thesis, Ohio State University Economics Department.

Liptser, Robert S., and Albert N. Shiryaev, 2001, *Statistics of Random Processes* (New York: Springer-Verlag).

Lo, Andrew, 1988, Maximum likelihood estimation of generalized Ito processes with discretely sampled data, *Econometric Theory* 4, 231–247.

Lucas, Robert E., 2000, Inflation and welfare, *Econometrica* 68, 247–274.

Marshall, David, 1992, Inflation and asset returns in a monetary economy, *Journal of Finance* 47, 1315–1342.

McCulloch, H., 1975, The tax-adjusted yield curve, *Journal of Finance* 30, 811–830.

McCulloch, H., 1990, U.S. term structure data, 1946-1987, in Friedman B.M., and F.H. Hahn, ed.: *Handbook of Monetary Economics*, vol. I, pp. 672–687. (North Holland, Amsterdam).

Mehra, R., and E. Prescott, 1985, The equity premium: A puzzle, *Journal of Monetary Economics* 15, 145–161.

Menzly, Lior, Tano Santos, and Pietro Veronesi, 2004, Understanding predictability, *Journal of Political Economy* 112, 1–47.

Nelson, D. B., 1990, ARCH models as diffusion approximations, *Journal of Econometrics* 45, 7–38.

Oksendal, Bernt, 2002, *Stochastic Differential Equations: An Introduction with Applications* (New York: Springer-Verlag).

Ravina, Enrichetta, 2005, Habit formation and keeping up with the Joneses: Evidence from micro data, Working paper, New York University.

Risa, Stefano, 2001, Nominal and inflation indexed yields: Separating expected inflation and inflation risk premia, Working paper, Columbia University.

Stambaugh, Robert, and Shmuel Kandel, 1991, Asset returns and intertemporal preferences, *Journal of Monetary Economics* 27, 39–71.

Stanton, Richard, 1997, A nonparametric model of the term structure dynamics and the market price of interest rate risk, *Journal of Finance* 52, 1973–2002.

Stock, J., and M. Watson, 1989, Does the monetary policy matter? A new test in the spirit of Friedman and Schwartz, in O. Blanchard, and S. Fisher, ed.: *Macroeconomic Annual*. (MIT Press, Cambridge, MA).

Sundaresan, Suresh, 1989, Intertemporal dependent preferences and the volatility of consumption and wealth, *Review of Financial Studies* 2, 73–88.

Veblen, Thorstein, 1994[1899], *The Theory of the Leisure Class*, (Allen & Unwin: London, 1899; reprinted by Dover Publications: New York).

Wachter, Jessica, 2005, Habit formation and returns on bonds and stocks, Working paper, Stern School of Business, New York University.

Wang, Jiang, 1996, The term structure of interest rates in a pure exchange economy with heterogeneous investors, *Journal of Financial Economics* 41, 75–110.

Wong, E., 1964, The construction of a class of stationary Markov processes, in R. Belleman, ed.: *Stochastic Processes in Mathematical Physics and Engineering*, Sixteenth Symposium in Applied Mathematics, pp. 264–276. (American Mathematical Society, Providence).