The Forward Valuation of Compound Options

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This article derives a solution for the valuation of compound options and American unprotected options when the underlying asset follows a diffusion process that is not a geometric Brownian motion. The solution is expressed as a forward integral of the price surface of European plain vanilla options. The result can be applied to price defaultable corporate coupon-paying bonds when the value of the common stock is modeled as an option on the value of the firm.

Apart from its theoretical interest, we show that the forward solution is significantly more efficient than alternative numerical methods to compute a cross-section of compound option prices and American options on dividend-paying stocks. Moreover, the forward solution can be directly used to build implied volatility trees starting from a cross-section of American options on dividend-paying stocks. This may be of particular interest, given that the great majority of traded options are American.

The majority of option pricing methods provide the price of options by a calculation that casts backward in time, starting from the maturity date. One exception is the forward partial differential equation of Dupire [1994], which provides the current prices of a panel of options as a function of their maturity dates and strike prices, treating as a given constant the current price of the underlying asset. Dupire's calculation proceeds forward from the knowledge of today's values of all options with immediate maturity.

An advantage of the forward pricing approach is that it is substantially more efficient than the backward pricing approach. The entire cross-section of European option prices over different maturity dates can be obtained numerically as the solution of just one no-arbitrage condition.[1]

Derman and Kani [1994] show how to use this forward representation to build implied trees. So far it has been believed that only European options could be priced by this method. In this article, we show that compound options can also be valued by a forward method, given the price surface of European options. This allows us to extend this approach directly to the case of American options.[2]

A compound option is an option written on an option. It is characterized by two maturity dates: the intermediate maturity, which is the date at which the buyer of the compound option may exercise the claim and receive a European option, and the final maturity, which is the maturity of the European option. At the intermediate date, the underlying asset price needs to be above a hurdle or strike price in order for the option to have positive value, which makes this kind of option a path-dependent derivative.

Since Geske [1979], several studies have focused on issues related to compound options. The reason for such interest is that a compound option is not just another option. It is the foundation for any option involving a sequence of exercise decisions, or "sequential options."[3]

The fact that compound or sequential optionalities are a pervasive feature of many financial contracts is well known. When a company is financially leveraged, common stocks are options on the value of the firm, and a *call option* on the stock is effectively a compound option on the value of the firm, as discussed in Geske [1979a].

The great majority of corporate bonds are callable by the issuer. Thus, because of the default risk, this embedded optionality is a compound option (a call on a put). This feature is particularly important for highly leveraged companies. Many convertible bonds are callable by the issuer. In this case, the compound optionality takes the form of a call on a call.

In the last ten years, one of the most successful corporate financing products has been the LYON. These are corporate zero-coupon bonds callable by the isssuer and convertible by the investor into common stock of the issuer. Given the interaction among the embedded optionalities, the value of a LYON is clearly not just the sum of the values of each embedded option. It is a compound option. Between April 1985 and December 1991, Merrill Lynch alone served as the underwriter for 43 separate LYON issues, which together raised a total of $11.7 billion for corporate clients.[4]

When a firm has coupon-paying bonds or bonds with early redemption features outstanding, Black and Scholes [1973] and Merton [1973] suggest that a *common stock* itself can be considered a compound option. At each coupon date, the stockholder can be thought as having the right to receive the next option, i.e., the common stock, by paying the coupon to the bondholders, i.e., the strike price. At the maturity date of the bond, the stockholder can reclaim the value of the firm by paying the face value of the debt. Prior to any coupon date, if the value of the firm is lower than the expected value of the debt, stockholders have a compound option that is out of the money, and they might find it optimal to exercise the right to abandon the compound option, i.e., to default.

According to this argument, corporate defaultable bonds can be valued as the difference between the value of the firm and the value of a compound option (see Geske [1977]). The closer the firm is to financial distress and the higher the dividend stream, the greater the difference between the valuation based on the compound option approach and standard methods.

Some of the most actively traded derivatives on the Chicago Board Options Exchange are American calls written against dividend-paying stocks. These call options are not "protected" against the price decline that is associated with the payment of dividends. This limits the possibility of using Merton's [1973] generalization of Black and Scholes [1973] that values options on stocks paying a continuous dividend flow.

Roll [1977] and Geske [1979b] price American call options on stocks with known dividends, showing that they can be replicated with a portfolio that includes European call options and a compound option. They use Geske's [1979a] solution for compound options to determine the quasi-analytic form for the American call. Whaley [1981] corrects their formula. In the same spirit, Geske and Johnson [1984] find a quasi-analytical formula expressed in terms of the sum of two infinite series of multinormal cumulative distribution functions. Selby and Hodges [1987] show how to improve the numerical efficiency of this approach by reducing the number of integrals to be evaluated.

It is widely recognized that many *capital budgeting problems* incorporate option features. This observation gave rise to the so-called real option literature. Compound options are very frequently encountered in capital budgeting problems when projects require sequential decisions:

- When dealing with development projects, the initial development expense allows one later to make a decision to wait or to engage in further development expenses eventually leading to a final capital investment project. All R&D expenditures involve a sequence of decisions.
- In the mining and extraction industries, one conducts geological surveys that will lead to the opening of a mine or to the decision to drill. Then, the owner of the mine or the drilling platform can any day stop operations and begin them again later. An investment in the production of a movie might lead to sequels. The value of a sequel is the value of a compound option.

The ability to derive the prices of compound and sequential options from those of plain European options would be valuable indeed.

I. WHICH STOCHASTIC PROCESS FOR THE UNDERLYING ASSET?

The Geske [1979a] formula to value compound options is based on the assumption of lognormality of the underlying asset. Yet, when the Black and Scholes [1973] pricing equation is used to extrapolate implied volatilities from observable European option prices, it has been shown that the volatility surface shows systematic changes across strikes and exercise dates. Beckers [1980] and Emanuel and MacBeth [1982] ask whether the reason for the departure from the Black and Scholes lognormality assumption is correlation between the level of the index and the volatility, as suggested by Cox and Ross [1976] and Cox [1996].[5]

Dupire [1994], Derman and Kani [1994], and Rubinstein [1994] extend this idea of a link between asset prices and their volatility. They generalize the Black-Scholes approach by considering volatility as a deterministic function of the level of the stock price and time to expiration. Dumas, Fleming, and Whaley [1998] investigate empirically the performance of deterministic volatility function models. Ait-Sahalia and Lo [1998] propose a non-parametric estimation procedure of the state price density implied in option prices. They find deviations from lognormality.

Why work with a model with generalized deterministic volatility as opposed to a fully stochastic volatility? Generalized deterministic volatility models are attractive for several reasons. First, they allow one to model time-varying volatility, and therefore to consider stochastic processes that are characterized by heteroscedasticity and leptokurtosis, properties that are frequently found in the empirical GARCH literature. Second, in these models, markets are dynamically complete. Thus, derivative securities can be priced by no-arbitrage without resorting to full-blown general equilibrium models and without the need to estimate risk premiums.[6]

Third, deterministic volatility models make it possible to fit the smile exactly by calibrating the local volatility function of the underlying asset. The attraction here is that it is possible to estimate state price densities that are exactly consistent with current observed option prices.

The empirical evidence on generalized deterministic volatility models is mixed. Bakshi, Cao, and Chen [1997] find that stochastic models outperform the simple Black and Scholes model. Jackwerth and Rubinstein [1998] run an extensive horse race among several stochastic volatility models and generalized deterministic volatility models. They find that some specifications in both classes outperform the simple Black and Scholes model in out-of-sample tests. Standard errors are too high, however, to conclude that stochastic models are any better or any worse than generalized deterministic volatility models.

Our main result is to give a forward representation of compound option prices for general diffusion processes with deterministic volatility. We show how to obtain the price of a compound option on the basis of pure European option prices, starting with today as an intermediate maturity date for which an initial condition is applied, and continuing the calculation to any actual intermediate maturity date. This is a "forward" calculation of the option price in the spirit of Dupire's [1994] forward equation for European options.

There is more than theoretical interest in this result. First, it gives a solution as a function of the strike price and the time to maturity, just as any broker would quote an option. This contrasts with the standard backward approach that gives a solution as a function of the level of the underlying asset and the current time.

Second, the forward solution can be used directly to generalize the Derman and Kani [1994] approach to build implied trees by forward induction. The advantage here is that calibration can be implemented using American options on dividend-paying stocks, one of the most actively traded contracts on the CBOE.[7] This can be particularly valuable since in the great majority of cases American options are the only contract that is available to calibrate an implied tree.[8]

From a practical perspective, the forward representation has the important numerical advantage that it gives an entire cross-section of compound options over several intermediate dates much more efficiently. We show that for 20 options the forward approach reduces computing time from 40 minutes to 11 seconds. Among other things, this lets us apply significantly more computing power to increased pricing accuracy.

II. THE FORWARD REPRESENTATION

Let $t = 0$ be the initial date, T_1 the intermediate-maturity date, and T_2 final one. We call K_1 the intermediate hurdle price and K_2 the exercise price of the second-stage European option. The underlying spot price at date 0 is called S_0, and the dummy variables for the prices of the underlying asset at date t are denoted by x.

Without any loss of generality, we modify Geske's problem somewhat by considering an option that would require no payment at the intermediate date. It is required only that the underlying asset price fall above some hurdle value in order for the option to continue in existence. The payment at the intermediate date can be valued as a piece of a plain European option, as described in Roll [1977] and Whaley [1981].

We denote by B_T the accumulated value of the cash account earning a continuously compounded interest equal to $r(t)$, $B_T = e^{\int_0^T r(u)du}$. Interest rates can be time-varying, and the term structure of zero-coupon bond prices is not restricted to assume any specific form. We do restrict interest rates to be deterministic, an assumption that would clearly be too restrictive if we were interested in pricing interest rate derivatives. In the case of equity options, however, Bakshi, Cao, and Chen [1997] find that models with stochastic interest rates do not significantly outperform the simple Black and Scholes model and underperform models with time-varying volatility.[9]

Under some technical conditions, Harrison and Kreps [1979] show that the absence of arbitrage opportunities is equivalent to the existence of a risk-neutral probability measure Q such that the price $\pi_t(X)$ of any tradable contingent claim with payoff X that settles at T is equal to $\pi_t(X) = B_t E_Q(B_T^{-1} X | \Omega_t)$.

We assume that, under the risk-neutral measure Q, the spot price of the underlying asset is:

$$dS(t) = S(t)\, r(t)\, dt + S(t)\, \sigma\left[S(t), t\right] \cdot dW(t) \quad (1)$$

where W is a d-dimensional standard Brownian motion under Q. Following Derman and Kani [1994], Dupire [1994], and Rubinstein [1994], we study economies in which the local volatility can be time-varying but restricted to be a bounded function of the price level of the underlying asset and time.

It is convenient to express the values of tradable securities in terms of a numeraire. Let the numeraire be the accumulated value of the cash account, and let us call $S^*(t) = B_T^{-1} S(t)$ the relative (or forward) price of the underlying asset and $X_T^* = B_T^{-1} X_T$ the discounted payoff.

Under the risk-neutral probability, the relative price $S^*(t)$ follows a (local) martingale:[10]

$$dS^*(t) = S^*(t)\, \sigma^*\left[S^*(t), t\right] \cdot dW_t \quad (2)$$

Let $q(S_0^*; x, t)$ be the risk-neutral probability density of a transition of the relative price from S_0^* at date 0 to x at date t. The relative price $\pi_t^*(X_T^*)$ of any tradable security can be obtained as the risk-neutral expected value of its future cash flows, expressed in units of the numeraire $\pi_t^*(X_T^*) = E_Q(X_T^* | \Omega_t)$. In the case of a compound option, its relative price Γ_0^* at the initial date is, therefore, equal to:

$$\Gamma_0^*\left(S_0^*, 0; K_1, K_2, T_1, T_2\right) = \int_{K_1^*}^{\infty} q(S_0^*; x, T_1) C^*(x, T_1; K_2, T_2)\, dx \quad (3)$$

where $C^*(x, T_1; K_2, T_2)$ is the relative price at date T_1 of a European call with strike price K_2 maturing on date T_2, conditional on the underlying asset being equal to x. The compound option is exercised when $S_{T_1} > K_1$, or $S_{T_1}^* \geq K_1^* \triangleq B_{T_1}^{-1} K_1$.

The two ingredients of the calculation below are the Fokker-Planck (forward) partial differential equation for the transition probabilities q and the (backward) Black-Scholes partial differential equation for the relative price $C^*(t)$ of the European call, both written at date t, $0 \leq t < T_1$. They are, respectively:

$$\frac{\partial}{\partial t} q\left(S_0^*; x, t\right) = \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\sigma^{*2}(x, t)\, x^2 q\left(S_0^*, x, t\right)\right] \quad (4)$$

$$-\frac{\partial}{\partial t} C^*(x, t; K_2, T_2) = \frac{1}{2}\sigma^{*2}(x, t)\, x^2 \frac{\partial^2}{\partial x^2} C^*(x, t; K_2, T_2) \quad (5)$$

$$\text{s.t. } C^*(x, T_2; K_2, T_2) = \max(0, x - K_2 B_{T_2}^{-1}) \quad (6)$$

We aim to obtain an ordinary differential equation for Γ_0^* as a function of the intermediate date of the compound option. Toward this end, we obtain the intermediate-maturity partial derivative:

$$\frac{\partial}{\partial t}\Gamma_0^*\left(S_0^*; K_1, K_2, t, T_2\right) = \int_{K_1^*}^{\infty} \frac{\partial}{\partial t} q\left(S_0^*; x, t\right) C^*(x, t; K_2, T_2)\, dx + \int_{K_1^*}^{\infty} q\left(S_0^*; x, t\right) \frac{\partial}{\partial t} C^*(x, t; K_2, T_2)\, dx \quad (7)$$

The first term of the right-hand side of this equation can be integrated by parts twice. We assume that the probability density goes to zero fast enough as x goes to infinity, so that all the boundary contributions at infinity are equal to zero.[11] This gives:

$$\frac{\partial}{\partial t}\Gamma^*_0\left(S^*_0; K_1, K_2, t, T_2\right) =$$

$$-\frac{\partial}{\partial K^*_1}\left[\frac{1}{2}\sigma^{*2}\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right] C^*\left(K^*_1, t; K_2, T_2\right) +$$

$$+\int_{K^*_1}^{\infty}\left[\frac{1}{2}\sigma^{*2}\left(x, t\right) x^2 q\left(S^*_0; x, t\right)\right]\frac{\partial^2}{\partial x^2} C^*\left(x, t; K_2, T_2\right) dx +$$

$$\left[\frac{1}{2}\sigma^{*2}\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right]\frac{\partial}{\partial K^*_1} C^*\left(K^*_1, t; K_2, T_2\right)$$

$$\int_{K^*_1}^{\infty} q\left(S^*_0; x, t\right)\frac{\partial}{\partial t} C^*\left(x, t; K_2, T_2\right) dx$$

Substituting (5), we get:

$$\frac{\partial}{\partial t}\Gamma^*_0\left(S^*_0; K_1, K_2, t, T_2\right) \quad =$$

$$-\frac{\partial}{\partial K^*_1}\left[\frac{1}{2}\sigma^2\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right] \times$$

$$C^*\left(K^*_1, t; K_2, T_2\right) +$$

$$\left[\frac{1}{2}\sigma^2\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right] \times$$

$$\frac{\partial}{\partial K^*_1} C^*\left(K^*_1, t; K_2, T_2\right) \tag{8}$$

This is an ordinary differential equation for Γ^*_0 as a function of the intermediate date of the compound option, which may be solved trivially by integration for any given S^*_0, K_1, K_2, T_2, with an initial condition:

$$\Gamma^*_0\left(S^*_0; K_1, K_2, 0, T_2\right) =$$

$$C^*\left(S^*_0, 0; K_2, T_2\right) \times \mathbf{1}\left(S^*_0 > K^*_1\right)$$

where **1** is an indicator function.

Observing that $\Gamma^*_0 = \Gamma_0$, $S^*_0 = S_0$, and $C^*_0 = C_0$, the final result is:

$$\Gamma_0\left(S_0; K_1, K_2, T_1, T_2\right) =$$

$$C\left(S_0, 0; K_2, T_2\right) \times \mathbf{1}\left(S_0 > K_1\right) +$$

$$\int_0^{T_1}\left\{-\frac{\partial}{\partial K^*_1}\left[\frac{1}{2}\sigma^{*2}\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right] \times\right.$$

$$C^*\left(K^*_1, t; K_2, T_2\right) +$$

$$\left[\frac{1}{2}\sigma^{*2}\left(K^*_1, t\right) K^{*2}_1 q\left(S^*_0; K^*_1, t\right)\right] \times$$

$$\left.\frac{\partial}{\partial K^*_1} C\left(K^*_1, t; K_2, T_2\right)\right\} dt \tag{9}$$

The price of a compound option is equal to the sum of three terms. The first term is the value at time 0 of a plain vanilla European option with strike equal to the final strike and maturity equal to the final date, truncated at the intermediate strike price of the compound option. The second term is a weighted average over the time dimension $0 \leq t \leq T_1$ of the prices of plain vanilla European options with strikes K_2, considered only for a value of the underlying asset equal to K^*_1. The third term is a weighted average of the strike sensitivities of these same plain vanilla European options. The parsimony of the European option price data input needed should be noted.

An advantage of any forward representation is that it gives a solution in the dual space of the backward approach. It expresses the price of derivative securities as functions of strike and time to maturity, given (S_0, t), just as any broker would quote these instruments. Since the forward representation gives the option surface up-front as it can be observed from market prices, its direct discretization can facilitate the construction of implied volatility trees based on securities with embedded compound options, such as American options. In the standard Black-Scholes backward approach, one would need a *full set* of backward solutions to obtain the market option surface.

III. THE FAMILY OF COMPOUND OPTIONS

Similarly, one can derive the prices of the entire family of compound options, i.e., a call on a put Γ^{cp}, a put on a call Γ^{pc}, and a put on a put Γ^{pp}. This is quite important since, for instance, a defaultable callable bond has an embedded call on a put option. Similarly, the LYONs are callable by the issuer and putable by the investor. Let $P(S_0, 0; K_2, T_2)$ be the price of a European put option at time 0.

Maintaining here the convention that compound options entail no payment at the intermediate date, one may observe that put-call parity relations hold as follows for pairs of compound options:

$$\Gamma_0^{cc} - \Gamma_0^{pc} = C\left(S_0, 0; K_2, T_2\right)$$

$$\Gamma_0^{cc} - \Gamma_0^{pc} = P\left(S_0, 0; K_2, T_2\right)$$

$$\Gamma_0^{cc} - \Gamma_0^{cp} = C\left(S_0, 0; K_1, T_1\right) + \left(K_1^* - K_2^*\right) \int_{K_1^*}^{\infty} q\left(S_0; x, t\right) dx$$

$$\Gamma_0^{pc} - \Gamma_0^{pp} = P\left(S_0, 0; K_1, T_1\right) - K_1^* - K_2^*) \int_{0}^{K_1^*} q\left(S_0; x, t\right) dx$$

IV. NUMERICAL ADVANTAGES OF THE FORWARD REPRESENTATION

Equation (9) may be used to provide in one fell swoop the prices of all compound options with different intermediate maturity dates T_1. Suppose you need to compute ten option prices with intermediate expiration $T_{1,j}$, $j = 1, ..., 10$. Given solutions for $q(S_0; K_1, t)$ and $C^*(K_1^*, t; K_2, T_2)$, which need to be computed in any approach, the marginal cost of generating option prices with intermediate dates $T_{1,j}$ with $j < 10$ is zero, if one uses the forward approach.[12] This is because solving for a compound option with intermediate date equal to 12 months necessarily implies also as a fall-out result the solution for the nine-month option and any other option with an earlier intermediate date.[13]

Given solutions for $q(S_0; K_1, t)$ and $C^*(K_1^*, t; K_2, T_2)$, if it takes one unit of time to compute $\Gamma_0(T_{1,10})$, then it would take ten units of time to compute the cross-section $\{\Gamma_0(S_0; K_1, K_2, T_{1,1}, T_2), \Gamma_0(S_0; K_1, K_2, T_1, 2, T_2), ..., \Gamma_0(S_0; K_1, K_2, T_{1,10}, T_2)\}$ using the backward approach. It does not take any additional time if one uses the forward approach.

An alternative way to compare the two methods is to study their relative performance in terms of pricing accuracy per unit of computing time. In Exhibit 1, we report the trade-off between computing time and pricing accuracy for cross-sections of compound options using the standard backward approach. Not surprisingly, the results show that the cost in terms of computing time of improving pricing accuracy grows non-linearly with the desired level of accuracy, proxied by the number of gridpoints.

EXHIBIT 1
Cost of Price Accuracy

Gridpoints	Number of Options 1	5	10	20	Marginal Cost
A. Backward Method					
50	0.91	2.54	4.63	8.94	0.42
100	2.82	11.84	22.10	42.52	2.08
200	12.36	58.09	118.65	223.00	11.09
400	71.16	347.80	716.00	1436.00	71.83
B. Forward Method—400 Options					
400	Time: 11.98 sec.				

Parameters are s = 0.2, S = 1, r = 0.02; final maturity date is $T_2 = 4$; and strike prices are $K_1 = 0.80$ and $K_2 = 1.20$. Cross-section is generated for different values of the intermediate date T_1. Marginal cost is the additional computing time to price one additional option. In the backward approach, the calculation is implemented on a three-dimensional grid (dS_1; dS_2; dT_1). In the case of only one compound option, computation time of the backward approach can be reduced by using an explicit two-dimensional grid. In the case of the forward approach, computations can always be implemented on a two-dimensional grid. Number of options refers to the number of contracts with common strike prices (K_1; K_2) and final date (T_2), but with different intermediate dates (T_1).

It takes 0.91 units of time to compute one compound option with 50 gridpoints on the price scale of the underlying asset both at the single intermediate-maturity date and at the final maturity date.[14] If one wants to improve the pricing accuracy by increasing the number of gridpoints to 400, computing time increases 78 times to 71.162 units of time.

At this level of price accuracy, computing a large cross-section of options is not practically feasible. It takes 24 minutes for a cross-section of 20 options and two hours for a cross-section of 100 options. The problem is even worse when one considers the case of multiple compounded optionalities, such as unprotected American options on dividend-paying stocks and real options.

One way to make the pricing of a cross-section of compounded options feasible is to reduce the level of pricing accuracy. Yet, we find that, in the lognormal case with $\sigma = 20\%$, the backward approach with 50 gridpoints generates prices that are off from the theoretical value by more than 1%. This is clearly not acceptable.

Using the forward method, a cross-section of 400 options can be priced at the highest level of pricing accuracy (400 gridpoints) in 11.98 seconds. The gain is by a factor of 601. The clear advantage of gaining computation time is to be able to invest significantly more computing power in improving pricing accuracy. In this case, the advantage is so great as to enable the pricing of a cross-section of the most actively traded contracts at the CBOE, such as an unprotected American call option on a dividend-paying stock. This can be achieved under assumptions on the process of the underlying asset that are less restrictive than lognormality.

V. EXTENSION TO MULTIPLE COMPOUNDING

Consider now a double-compound option with two hurdle prices, K_1 placed at date T_1 and K_2 placed at date T_2, while the final option expiration is at $T_3(0 \leq T_1 \leq T_2$ K $T_3)$ with a strike price equal to K_3. After determining the prices $\Gamma(K_1, K_2, K_3, t, T_2, T_3)$ of simple compound options at dates $t(0 \leq t \leq T_1)$ as a function of the prevailing asset price K_1, using Equation (9), it is a simple matter to obtain the prices $\Upsilon_0(S_0, K_1, K_2, K_3, T_1, T_2, T_3)$ at date 0 of double-compound options.

They are given by the formula:

$$\begin{aligned}
&\Upsilon_0\left(S_0, K_1, K_2, K_3, T_1, T_2, T_3\right) = \\
&\quad \Gamma_0\left(S_0; K_2, K_3, T_2, T_3\right) \times \mathbf{1}\left(S_0 > K_1\right) + \\
&\quad \int_0^{T_1} \left\{ -\frac{\partial}{\partial K_1^*}\left[\frac{1}{2}\sigma^{*2}\left(K_1^*, t\right) K_1^{2*} q\left(S_0; K_1^*, t\right)\right] \times \right. \\
&\quad \Gamma^*\left(K_1^*, t; K_2, K_3, T_2, T_3\right) + \\
&\quad \left[\frac{1}{2}\sigma^{*2}\left(K_1^*, t\right) K_1^{*2} q\left(S_0; K_1^*, t\right)\right] \times \\
&\quad \left. \frac{\partial}{\partial K_1^*}\Gamma^*\left(K_1^*, t; K_2, K_3, T_2, T_3\right) \right\} dt \qquad (10)
\end{aligned}$$

Equation (10) can be readily extended to a general sequence of compounding optionalities, so that the result can be applied to price instruments such as corporate defaultable coupon-paying bonds, callable corporate bonds, callable convertibles, or LYONs.[15] Other applications are to capital budgeting problems when projects require sequential decisions.

VI. APPLICATION TO UNPROTECTED AMERICAN OPTIONS AND TO SEQUENTIAL DECISIONS

One of the most actively traded contracts on the CBOE is an American option on a dividend-paying stock. Let αD be the unprotected decline in the value of the underlying asset after the payment of a dividend equal to D at date T_1. Roll [1977] and Geske [1979b] discuss the possibility of replicating this derivative using compound options. The same idea applies to the many times sequential exercise decisions must be made.

Let $\hat{S}(T_1)$ be the critical level of the underlying asset price above which the American call is exercised early. It is defined implicitly as the solution of:

$$C(\hat{S}(T_1), T_1; K, T_2) = \hat{S}(T_1) + \alpha D - K \qquad (11)$$

where K is now the single strike of the American-type option, and T_2 is its maturity date $(0 \leq T_1 \leq T_2)$. When the date T_1 asset price is below $\hat{S}(T_1)$, the continuation value is higher than the exercise value.[16]

Whaley [1981] shows that a replication strategy as follows is equivalent to the American option:

1. A long position in a European call option with exercise price K_2 and maturity date T_2.
2. A long position in a European call option with exercise price $\hat{S}(T_1)$ and maturity date $T_1 - \varepsilon$, with $\varepsilon \cong 0$.
3. A short position in one European *compound option* with intermediate date $T_1 - \varepsilon$, maturity date T_2, and strike price $\hat{S}(t) + \alpha D - K_2$.

Hence, the price c of the American option is:

$$\begin{aligned}
c(S_0, 0; K_2, T_2) = {} & C(S_0, 0; K_2, T_2) + \\
& C(S_0, 0; \hat{S}(T_1), T_1 - \varepsilon) - \\
& \Gamma_0(S_0, \hat{S}(T_1) + \alpha D - K_2, K_2, T_1, T_2)
\end{aligned}$$

Here Γ is, of course, a complete compound option, featuring not simply an interim hurdle price but with an actual disbursement occurring at the interim date. Hence Equation (9) will have to be amended by addition of the appropriate part of the European option value that reflects the disbursement.

When the underlying asset follows a general diffusion process with non-constant volatility, the results of Geske [1979a] are no longer applicable. In this more general case, $\Gamma(S_0, \hat{S}(T_1) + \alpha D - K_2, K_2, T_1, T_2)$ can be valued using our results so far. This provides a method to compute the entire surface of American unprotected options by solving just one ordinary differential equation. The method is applicable to options whose lives are long enough to cover several dividend payments.

VII. A SPECIAL CASE OF EQUATION (9)

In the special case when the underlying asset follows a geometric Brownian motion, Geske [1979a] provides an explicit formula for the price of a compound option. Under this assumption, $\sigma(S, t)$ is constant, and $q(S_0; K_1, u)$ is a lognormal probability density function. If we substitute these conditions in Equation (9), we can verify numerically that the forward pricing formula provides the same values as obtained by Geske [1979a]:

$$\begin{aligned}\Gamma_0\left(S_0, K_1, K_2, T_1, T_2\right) \;=\;& \\ & S_0 N_2\left(a_1, b_1, \sqrt{\tfrac{T_1}{T_2}}\right) - \\ & e^{-rT_2} K_2 N_2\left(a_2, b_2, \sqrt{\tfrac{T_1}{T_2}}\right) - \\ & K_1 e^{-rT_1} N(-u) \end{aligned} \qquad (12)$$

where u is the critical value of the standardized Gaussian variable $1/\sqrt{T_1}\, W_{T_1}$ at which the compound option is at the money. The value of u is obtained numerically as a solution to:

$$C(S_0 e^{(r-\frac{1}{2}\sigma^2)T_1 + u\sigma\sqrt{T_1}}, T_1; K_2, T_2) = K_1$$

where N_2 is the cumulative bivariate unit normal density function and:

$$a_1 \;=\; -u + \sigma\sqrt{T_1}; \qquad a_2 = a_1 - \sigma\sqrt{T_1}$$

$$b_1 \;=\; \frac{\ln\left(\frac{S_0}{K_2}\right) + (r - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}; \quad b_2 = b_1 - \sigma\sqrt{t}$$

A numerical comparison of (9) and (12) is given in Exhibit 2. The results confirm that under the hypothesis of lognormality, the forward approach gives the same results as Geske's quasi-closed-form solution. The results in Exhibit 2 are obtained in the case of a constant interest rate equal to 3%. We obtain similar results also in the case of non-monotonic (but deterministic) term structures of interest rates.

EXHIBIT 2
Numerical Comparison of Forward Pricing Approach and Geske [1979a]

		Final Strike K_2				
		0.8			1.2	
K_1	T_1	Buraschi-Dumas	Geske	T_1	Buraschi-Dumas	Geske
	0.2	2.319198725e-01	2.319198269e-01	0.2	2.766162972e-02	2.766154758e-02
0.8	0.5	2.305935972e-01	2.305935566e-01	0.5	2.766328463e-02	2.766319294e-02
	0.8	2.310779750e-01	2.310779611e-01	0.8	2.766559521e-02	2.766557534e-02
	0.2	1.407648569e-01	1.407648124e-01	0.2	2.171108727e-02	2.171100807e-02
1.01	0.5	1.653012215e-01	1.653011804e-01	0.5	2.610673320e-02	2.610672741e-02
	0.8	1.819052978e-01	1.819052374e-01	0.8	2.758595645e-02	2.758593657e-02
	0.2	1.002052629e-02	1.002043411e-02	0.2	2.749191394e-03	2.749191394e-03
1.2	0.5	5.150156542e-02	5.150153013e-02	0.5	1.427979118e-02	1.427979019e-02
	0.8	8.510490284e-02	8.510490131e-02	0.8	2.334733232e-02	2.334728781e-02

Parameters are $\sigma = 0.2$, $S = 1$, $r = 0.03$; final maturity date is $T_1 = 1$. K_1 is strike price associated with intermediate-maturity date T_1; K_2 is the final strike price associated with final maturity T_2.

VIII. CONCLUSIONS

When the Black and Scholes [1973] pricing equation is used to extrapolate implied volatilities from observable European option prices, several studies have found that the volatility surface changes systematically across strikes and exercise dates. Some suggest that one reason for the deviations is that stock prices are not lognormally distributed and the variance is not constant. If this is the case, we need numerically feasible solutions for the value of sequential derivatives such as compound options and traded American options.

We have shown, using forward arguments, that the value of a compound option can be computed in terms of the price surface of European options. An advantage of this result over the backward approach is that it can be used to provide at one time the prices of all compound options with different intermediate-maturity dates and strike prices. Since American unprotected options—as well as most sequential decision-making situations—can be replicated by a compound option and European options, we can apply our result to price common actively traded contracts under general assumptions on the diffusion process.

ENDNOTES

[1]Numerical implementation of the backward pricing approach requires instead solving for as many arbitrage conditions as option contracts, indexed by their strike prices.

[2]In the process of building implied trees, the forward solution becomes important because the method requires a very dense price surface over all strike prices (nodes) and times to maturity.

[3]We explain later the link between compound options and sequential options in detail later .

[4]See for a discussion of LYONs McConnell and Schwartz [1986]. Notable issuers of LYONs are American Airlines, Eastman Kodak, Marriott Corporation, and Motorola. Euro-Disney raised $965 million with a LYON issue in June 1990. As the *Wall Street Journal* put it, this fixed-income product with compounded optionalities turned out to be "one of Wall Street's hottest and most lucrative finance products" (December 17, 1991, p. C1).

[5]When a firm is financially leveraged, if the value of the firm follows a lognormal process, the value of the stock cannot be lognormally distributed. Given the value of the firm, exogenous shocks to the equity value affect the debt-to-equity ratio, and thereby the riskiness of the firm's stock and its variance, which contradicts the assumption of lognormality.

[6]The estimation of risk premiums can be highly problematic, especially in the case of high-frequency data, since it is difficult to measure the intertemporal marginal rate of substitution. Moreover, any pricing result would be sensitive to assumptions on the preferences of the economic agents.

[7]The literature has focused on building implied trees using European plain vanilla options.

[8]An additional advantage is that it allows one to take into account the fact that for highly leveraged companies a European option is effectively a compound option on the value of the assets of the firm.

[9]The result makes good economic sense if one compares the numerical values of ρ and σ for equity options and the relative volatility of interest rates and equities. For a one-year at-the-money option, with 25% and interest rates r = 5.8%, the impact of a one-standard deviation change in interest rates is 6.5 times less than a one-standard deviation change in the price of the underlying asset.

[10]With no loss of generality, when interest rates are deterministic we can reparameterize the local volatility as a function of the forward price, so that $\sigma^*(S_t^*, t) = \sigma(B_t S_t^*, t)$.

[11]The two assumptions that we refer to are as follows:

$$\lim_{x\to\infty} \frac{\partial}{\partial x}\left[\frac{1}{2}\sigma^{*2}(x,t)x^2 p(S_o^*;x,t)\right] C^*(x,t;K_2,T_2) = 0$$

and

$$\lim_{x\to\infty} \left[\frac{1}{2}\sigma^{*2}(x,t)x^2 p(S_o^*;x,t)\right] \frac{\partial}{\partial x} C^*(x,t;K_2,T_2) = 0$$

The Black and Scholes [1973] model is an example of an economy that satisfies these two assumptions. In this case, as the underlying goes to infinity, the value of the call option goes to infinity, and the value of the delta converges to one. The probability density $p(S_0^*; x, t)$ converges to zero at an exponential rate, however, so that both these restrictions are satisfied.

[12]These solutions can be obtained for general diffusion processes using well-known methods such as the finite-difference approach or Monte Carlo.

[13]The forward method, by its very nature, provides the theta and strike sensitivity (kappa) but not the delta and gamma. And neither the forward nor the backward method would provide the vega.

[14]On a Pentium Pro 200, the units of time are seconds. The results are obtained by means of a Matlab code that is available upon request.

[15]Merton [1973], Geske [1977].

[16]When the stock goes ex-dividend and no additional dividends are expected before the maturity of the option, the value of the American option is identical to the value of the European counterpart with the same strike.

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