ARTICLE IN PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Financial Economics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

JOURNAL OF Financial ECONOMICS

www.elsevier.com/locate/econbase

# Inflation risk premia and the expectations hypothesis☆

Andrea Buraschi[a,b,*], Alexei Jiltsov[c]

[a]*London Business School, Sussex Place, Regents Park, London NW1 4SA, UK*
[b]*CEPR*
[c]*Lehman Brothers, London, UK*

Received 21 August 2003; received in revised form 7 October 2003; accepted 4 November 2004

**Abstract**

We study the properties of the nominal and real risk premia of the term structure of interest rates. We develop and solve the bond pricing implications of a structural monetary version of a real business cycle model, with taxes and endogenous monetary policy. We show the relation of this model with the class of essentially affine models that incorporate an endogenous state-dependent market price of risk. We characterize and estimate the inflation risk premium and find that over the last 40 years the ten-year inflation risk premium has been has averaged 70 basis points. It is time-varying, ranging from 20 to 140 basis points over the business cycle and its term structure is sharply upward sloping. The inflation risk premium explains 23% (42%) of the time variation in the five (ten)-year forward risk premium and

☆The authors want to thank Lars P. Hansen for several useful comments and suggestions on an earlier version of the paper, and Geert Bekaert, David Chapman, John Cochrane, George Constantinides, Ian Cooper, Silverio Foresi, Stefano Risa, Nizar Touzi, Raman Uppal, and seminar participants at Carnegie Mellon, The University of Chicago, Duke University, Insead, London Business School, University of Michigan, MIT Sloan School, Nortwestern University, UCLA, University of Rochester, Wharton, and the WFA Conference. Special thanks to the WFA Committee which awarded this paper the Best Paper Award in Investment.

*Corresponding author. London Business School, Institute of Finance, Sussex Place, Regents Park, London NW1 4SA, UK. Tel.: +44-207-262-5050; fax: +44-207-724-3317.

*E-mail address:* aburaschi@london.edu (A. Buraschi).


doi:10.1016/j.jfineco.2004.07.003

it plays an important role in help explain deviations from the expectations hypothesis of interest rates.


*JEL classification:* D9; E3; E4; G12

*Keywords:* Term structure; Bond prices; Inflation; Expectations hypothesis; Risk premium

## 1. Introduction

An enormous literature discusses the estimation and properties of dynamic models of the term structure of interest rates. A significant part of this literature focuses on reduced-form affine models.[1] In this paper, first we study the link between these models and the real business cycle literature. We then quantify the properties of the inflation risk premium and study the ability of a general equilibrium model to explain deviations from the expectations hypothesis of interest rates.

The first generation of (completely) affine models makes three assumptions to derive implications about the nominal yield curve. First, the spot interest rate is an affine function of a set of mean-reverting state variables with constant or square-root local volatility. Second, the price of risk is a constant multiple of the local interest rate volatility. Third, inflation is neutral so that the Fisher relation between nominal and real interest rates holds.

Empirical studies of this class of models have exposed several limitations. With regards to the second assumption, Duffee (2002) shows that the restriction on the market price of risk implies bond returns and Sharpe ratios that are too high with respect to the empirical evidence. Dai and Singleton (2000) show that this same assumption makes affine models unable to explain the extent of the deviation from the expectations hypothesis of interest rates. They state that "a three factor CIR-style [Cox, Ingersoll, and Ross] model is wholly incapable of matching linear projection yield ($LPY$) coefficients. We attribute this model failure to the constraint in CIR-style models that risk premiums are proportional to factor volatilities". With regards to the third assumption, there is mounting evidence against the Fisher neutrality assumption. Benninga and Protopapadakis (1983), Fama (1990), and Boudoukh (1993) find that the inflation rate is negatively related to the real interest rate in terms of both realized changes and expected values. Moreover, real returns on nominal bonds decline when inflation increases (Fama, 1976b, 1990; Fama and Gibbons, 1982).[2]

[1]Important studies of nonlinear models include Naik and Lee (1997), who introduce Markov regime switching into an affine model, and Bansal and Zhou (2002) who study Markov switching in the context of richer discrete-time affine models.

[2]Fama (1979) also finds evidence of inflation nonneutrality in the stock market and shows that stock real returns are negatively correlated with inflation. In the medium and long term, the real gross domestic product is negatively affected by an increase in inflation (Fama and Gibbons, 1982; Boudoukh, 1993; Harvey, 1988).

Partially in response to these limitations, recent studies have explored more flexible models. Duffee (2002) and Dai and Singleton (2000) estimate reduced-form (essentially) affine models in which the price of risk is specified as a more general (ad hoc) function of the state variables. They identify and discuss the specific features that improve the empirical performance of this class of models.

In this paper, instead of assuming these features as exogenous, we develop and estimate a structural model in which some of these features arise in equilibrium. We explore a monetary version of a real business cycle production economy in which, in equilibrium, the term structure of interest rates has the following properties. First, although the state variables follow affine stochastic processes, the market price of risk is not a constant multiple of the local volatility of interest rates. Second, the inflation risk premium is positive and time varying, i.e., the Fisher hypothesis does not hold. Third, the previous two features make the term structure deviate from the expectations hypothesis of interest rates. Thus, the model can potentially match the empirical LPY coefficients.

The structural model allows us to identify the underlying nominal and real factors and to address a number of economic questions. However, this approach has its shortcomings. First, it shares the known limitations of real business cycle models (Cooley and Hansen, 1995). Second, while the market price of risk is state-dependent, it is not as flexible as the one advocated in the reduced-form approach by Duffee (2002). We study the extent to which the model can describe the dynamics of the term structure despite these limitations. Our structural model is classical in many respects: time-separable preferences, a representative agent, diffusive information, and a constant-returns-to-scale production function. Its main distinguishing features are as follows.

First, we assume a nominal fiscal system. This feature generates the departure from the Fisher hypothesis. In classical monetary real business cycle models, inflation and money are not neutral because if agents anticipate an increase in inflation, they substitute away from activities that use cash in favor of activities that do not. We explore a different and arguably more important channel of monetary nonneutrality. When the fiscal system is not indexed to the general price level, i.e., when taxes and fiscal incentives are calculated on nominal historical values, the inflation rate affects the after-tax real return on capital. This, in turn, affects ex ante decisions on the optimal allocation of (real) resources and therefore also affects both asset prices and risk premia (see Feldstein et al., 1978; Feldstein, 1980; Fisher and Modigliani, 1978). Examples of the nominal nature of the fiscal system include depreciation, capital gains, and interest payments on debt.[3] When the fiscal system is

[3]Depreciation is usually computed on the nominal ‘‘historical’’ cost of the assets, rather than on the ‘‘current’’ nominal replacement value. Thus, an increase in expected inflation reduces the real after-tax return on capital, which reduces the optimal level of long-lived capital investment. In turn, this affects the equilibrium term structure of interest rates. Capital gains are taxed according to the nominal value of the capital invested. Thus, an increase in inflation reduces the after-tax return on equity, given the pre-tax real return. This affects the optimal capital structure favoring forms of debt financing. Interest payments on debt are usually deductible in terms of their nominal value. Thus, an increase in inflation reduces the after-tax cost of debt financing.

based on nominal historical values, inflation is a risk factor with asset pricing implications. Given the dimension of the taxable base and tax rates, the fiscal system is of first-order importance.

Second, monetary policy is endogenous. Similar in spirit to a Taylor (1993) policy rule, the money supply consists of a constant long-term rate plus a term that depends on the gap between the current levels of inflation and output and their long-term targets. This feature allows us to distinguish between exogenous monetary shocks and monetary injections motivated by real shocks.

Third, the real marginal productivity of capital follows a square-root process with stochastic drift. The state variables affecting the drift follow square-root processes with correlated Brownian motions. This setup can capture differential effects of the state variables on the marginal productivity of capital. Under certain parameter configurations, the state variables can independently affect either the instantaneous return on capital or its local variance. This allows us to model separately the uncertainty (unexpected innovations) in the marginal productivity of capital and the volatility of expected innovations in the marginal productivity of capital. We show the importance of this feature to generating a state-dependent market price of risk.

We begin by fully characterizing the stochastic equilibrium of the model. We then estimate the structural parameters using U.S. Treasury bond data and study a number of economic implications. We address several questions. First, can a classical monetary business cycle model generate an affine term structure with a price of risk sufficiently flexible to address Duffee's critique? We show a model that generates an endogenous equilibrium market price of risk that is not a constant multiple of the interest rate volatility. The price of risk is state-dependent and can explain the conditional volatility of interest rates better than a traditional CIR model.

Second, what is the size of the U.S. inflation risk premium? We find that over the last 40 years, the average one-month inflation risk premium has been 15 basis points. However, the average ten-year inflation risk premium has been 70 basis points. The term structure of the inflation risk premium is sharply upward sloping, with the long-term inflation risk premium about four times larger than the short-term premium. The size of the long-term inflation risk premium is a large component of the yield spread between nominal and real bonds. Moreover, the inflation risk premium shows time variation over the business cycle, from 20 to 140 basis points.

Third, can a structural model explain the size of the deviation from the expectations hypothesis (EH) of interest rates? What structural reasons drive such deviations? Our monetary model generates a highly time-varying forward risk premium. The extent of time-variation is sufficient to reject the EH. We analytically solve for the model-implied Campbell-Shiller (1991) regression coefficients. We find that they are not statistically different from those obtained by Campbell and Shiller using empirical data. We then decompose the total risk premium into two components. The first is generated by monetary shocks and the second by real shocks. We find that the monetary factor accounts for 43% of the time variation of the risk premium.

This paper draws on contributions from several streams of literature. In addition to the reduced-form affine term structure literature, discussed earlier, our asset pricing model is closely related to monetary versions of real business cycle models.[4] As such, our model inherits many of their advantages and disadvantages. For instance, in the absence of frictions, time-separable real business cycle models find it difficult to replicate output growth persistence (Cooley and Hansen, 1995) and the equity premium. For this reason, classical real business cycle models have been generalized to incorporate nominal rigidities, more realistic financial intermediation mechanisms, and nonseparable preferences. Blanchard and Kiyotaki (1987) study a static economy with both wage and price stickiness. Chari et al. (1996) consider a more general real business cycle model with price stickiness. Erceg (1997), Erceg et al. (2000), and Huang and Liu (1999) analyze the effects of exogenous monetary policy shocks in a model with wage contracts à la Taylor. Christiano et al. (2001) consider a model with staggered wage contracts and variable capital utilization. They show that these nominal rigidities can account for observed inertia in inflation and persistence in output. Although they do not derive explicit implications for the inflation risk premium, it is reasonable to expect that these nominal rigidities would increase the inflation risk premium, implied by models with no frictions, by distorting the optimal allocation of capital and increasing the persistence of monetary and technological shocks.

Cooley and Nam (1998) study a different monetary channel. They incorporate a debt-contracting problem with costly verification into a standard real business cycle model with limited participation. Financial intermediaries are initially uninformed and must pay a price to obtain information. This feature affects loan intermediation and amplifies the response of capital to the money supply shock.[5]

Our work is also related to Bakshi and Chen (1996), who study a monetary economy in which positive monetary holdings are supported in equilibrium. They derive closed-form solutions for the nominal term structure of interest rates in a model that abstracts from tax distortions. The main differences in our model are: (a) the monetary policy is endogenous (the nominal money supply is allowed to change in response to deviations from monetary and real targets); (b) we introduce taxes in the model and allow for an imperfect indexation mechanism to nominal shocks; and (c) we let the investment opportunity set be affected by inflation innovations so that there exists a (time-varying) risk premium on the inflation rate. The equilibrium process of the general price level is affected by both supply and demand factors. Our results are also related to Evans (1998), who uses data on U.K. index-linked bonds to estimate the real term structure and the risk premium, and to Pennacchi (1991), who

[4]Danthine et al. (1987) consider a stochastic production economy and explore whether Sidrauski's money superneutrality result holds in a monetary equilibrium with uncertainty. They do not solve the model in closed form but show that money is nonneutral.

[5]Another important contribution include Fuerst (1995) who extends the cash-in-advance model by incorporating a real financial sector and agency costs into the Christiano and Eichenbaum monetary model. This captures the notion of a "credit channel" as in Bernanke and Gertler (1989). In that paper the condition of borrowers' balance sheets is a source of output dynamics. Higher borrower net worth reduces the agency costs of financing real capital investments. Thus, shocks that affect net worth can initiate fluctuations. Other papers that extend the classical real business cycle model to account for financial intermediation include Fisher (1999), Carlstrom and Fuerst (1998), and Bernanke et al. (2000).

studies a generalized Vasiček economy with constant volatility in which the time series for the expected inflation is obtained from survey data as opposed to being estimated. An important advantage of our model is to allow the risk premia to be time varying.[6] Other related papers include Longstaff and Schwartz (1992), Marshall (1992), the two-factor Cox et al. (1985b) model, Constantinides (1992), and Duffie and Kan (1996). However, none of these papers explicitly account for the risk premium on the inflation rate.

The paper is organized as follows. Section 2 sets up the model and characterizes the equilibrium nominal and real term structures of interest rates. Section 3 describes empirical testable restrictions and the econometric method. Section 4 describes the dataset. Section 5 summarizes the results of estimating and testing the structural model. Section 6 illustrates the properties of the inflation risk premium. Section 7 explores the extent to which the model can explain deviations from the expectations hypothesis. Section 8 discusses the implications of the model in terms of conditional second moments. Section 9 studies the tradeoff between fitting asset prices and other macroeconomic variables. Section 10 concludes.

## 2. The structure of the economy

We consider an economy in which a single good is produced by a representative agent who can either consume or reinvest it in a constant-returns-to-scale production technology. Real monetary holdings are assumed to provide a transaction service because they reduce the total amount of resources needed to achieve a given level of net consumption. In this economy, money is held because of its positive marginal productivity in the "shopping technology". Let the preferences $U(t, X_t)$ of the representative agent be time separable and logarithmic in real net consumption holdings $X$, i.e., $U(X) = \int_0^\infty e^{-\rho t} \ln X_t \, dt$.

**Assumption 1.** Real monetary holdings $M_t^d$ provide a transaction service because they reduce the total amount of gross resources $X_t$ needed to obtain a given level of net consumption $C_t$. We model this feature by including money in the utility function

$$X_t = C_t (M_t^d)^\gamma \quad \text{with } 0 \leqslant \gamma \leqslant 1. \tag{A1}$$

The service provided by monetary holdings has decreasing returns to scale, so that $\frac{\partial^2 X_t}{\partial M^2} < 0$. When $\gamma = 0$, monetary holdings do not generate any transaction service.

The gross rate of return on capital depends on multiple technological shocks. Part of the total output is absorbed by the public sector which levies taxes on nominal profits at a rate $t$. The remaining net capital is optimally allocated to consumption and real monetary holdings and/or reinvested. We do not model public expenditure,

[6] Pennacchi (1991) estimates a homoskedastic VAR of instantaneous real interest rates and expected inflation based on a generalization of Vasič ek model. The generality of his approach is limited by the homoskedasticity assumption which implies time-invariant risk premia.

which could introduce additional distortions. We assume that public expenditure does not affect the distribution of wealth and instead focus on the pricing implication of the tax system (an example would be the case in which the public expenditure is distributed in lump sums).

**Assumption 2.** The real after-tax capital accumulation process depends on a $k$-dimensional vector of technological shocks $\mathbf{Y}_t$. The production technology cost structure consists of depreciation (maintenance cost) and a variable cost proportional to output. The depreciation rate is assumed to be $\lambda_m$, the variable cost $\lambda_s$, and the lump-sum transfers $\ell$. The real after-tax return on capital can be allocated to consumption $C_t$, or real monetary holdings $M_t^d$ and/or reinvested, $\mathrm{d}K_t$:

$$\begin{aligned} C_t\,\mathrm{d}t + M_t^d\,\mathrm{d}t = {} & \underbrace{K_t\mathbf{1}'\,\mathrm{d}\mathbf{Y}_t}_{\text{total output}} - \underbrace{\lambda_m K_t\,\mathrm{d}t}_{\text{depreciation}} - \underbrace{\lambda_s K_t\mathbf{1}'\,\mathrm{d}\mathbf{Y}_t}_{\text{variable production cost}} \\ & - \underbrace{T_1 - T_2}_{\text{taxes}} + \underbrace{\ell K_t\,\mathrm{d}t}_{\text{lump-sum transfers}} - \underbrace{\mathrm{d}K_t}_{\text{investment}}. \end{aligned} \tag{A2}$$

The entrepreneur has access to a real storage technology and we study the extent of the welfare costs of inflation and the size of the inflation risk premium due to the nominal nature of the fiscal system. If the storage technology were nominal, the distortionary effect of inflation would be even larger. The fiscal system is described as follows:

**Assumption 3.** The fiscal authorities impose taxes on operating income and capital gains. All costs of the entrepreneur are tax-deductible. However, as in the U.S. tax code, we assume that the tax shield is calculated with respect to the historical nominal value of any capital expenditures. The real value, as of time $t+h$, of the income tax liability is therefore equal to

$$T_1 = \lim_{h\to 0} \tau_{\mathrm{pr}} \underbrace{\left(K_t\mathbf{1}'(\mathbf{Y}_{t+h} - \mathbf{Y}_t) - \frac{p_t}{p_{t+h}}\lambda_m K_t h - \frac{p_t}{p_{t+h}}\lambda_s K_t \mathbf{1}'(\mathbf{Y}_{t+h} - \mathbf{Y}_t)\right)}_{\text{taxable income with expenses deducted at historical cost}}. \tag{A3}$$

The capital gains tax is equal to

$$T_2 = \lim_{h\to 0} \underbrace{\tau_{\mathrm{cg}}\left(\frac{p_{t+h} - p_t}{p_{t+h}}\right)K_t}_{\text{capital gains tax}}.$$

Taxes are redistributed via a lump-sum linear subsidy $\ell K_t\,\mathrm{d}t$.

Assumption (A3) describes the taxation of operating income, $T_1$, and capital gains, $T_2$. The value of depreciation allowances is based on the nominal "historical" cost of the assets rather than on their "current" nominal replacement value. Thus, an increase in price levels decreases the real value of depreciation so that the real value of taxable profits rises. The effect is larger for firms using the longest-lived capital. Through this channel, an increase in the expected inflation rate induces a decrease in the rate of investment and a shift to shorter-lived capital. Thus, inflation is a priced risk factor. The second channel that makes inflation priced is the taxation of nominal capital gains. Given a constant real pre-tax return on investment, an increase in the inflation rate decreases the after-tax real return. Thus, under a nominal fiscal system, inflation affects the equilibrium real marginal rate of transformation and therefore asset prices.

We assume a linear tax liability. This has some limitations. Firstly, when a company is not in a tax-paying position, negative tax liabilities are carried forward to future periods. Hence, the value of a negative tax liability should be expressed as the discounted value of the carryforwards. Second, it implicitly rules out a tax-timing option, as discussed in Green and Hollifield (2003). Including the tax-timing option, however, would be computationally formidable since its optimal exercise policy is path dependent, due to the tax base evolution. Further, the linear specification of the tax structure is necessary to have a constant-return-to-scale production technology, which is needed to obtain closed-form solutions.

The total amount of taxes collected is either distributed via lump-sum transfers $\ell K_t\,\mathrm{d}t$ to the private sector or absorbed by the public sector. This second use of resources represents a deadweight loss for the productive side of the economy. We do not model the fiscal policy of the public sector and simply take the tax and transfer rate as exogenous parameters.

If we substitute (A3) into (A2), given an equilibrium price $\frac{\mathrm{d}p_t}{p_t}$, we obtain the capital accumulation process. Inflation affects the after-tax return on investment in several ways. First, given a positive capital gains tax $\tau_{\mathrm{cg}}$, higher inflation rates lead to higher real values of the tax liabilities due to nominal capital appreciation. Thus, the higher the inflation rate, the higher is the opportunity cost of the investment. Second, the higher the inflation rate, the lower is the real value of costs deducted for tax reasons and of depreciated capital. These effects make entrepreneurs using capital-intensive technology more averse to inflation shocks. Since $\tau_{\mathrm{cg}}$ is about 30% in the U.S., and higher in other countries, inflation has a first-order effect on the real accumulation of capital and therefore on equilibrium asset prices.

Indexation of the fiscal system reduces the relative size of the inflation tax.[7] When a fiscal system is perfectly indexed to the general price level, taxes become a function of *real* operating income. The capital accumulation process simplifies to

$$\frac{\mathrm{d}K_t}{K_t} = \left[ -\frac{C_t}{K_t}\,\mathrm{d}t + \mathrm{d}Y_t(1-\tau_{\mathrm{pr}})(1-\lambda_s) - [\lambda_m(1-\tau_{\mathrm{pr}}) - \ell]\,\mathrm{d}t - \frac{M_t^d}{K_t}\,\mathrm{d}t \right].$$

Even though nonneutrality of nominal shocks has been debated in macroeconomics for a long time, empirical models of the term structure usually assume that the Fisher hypothesis holds.[8] Cox et al. (1985b) have two models with an exogenous

[7]Several countries have recently tried to limit the nominal nature of their fiscal system by introducing different forms of indexation.

[8]An important exception is Pennacchi (1991), who generalizes the Vasiček model to distinguish between real and nominal interest rates and uses survey data to identify inflationary expectations. Since in his Vasiček-type model the factors are Ornstein-Uhlenbeck processes, his model is exposed to the usual criticism that nominal and real interest rates can become negative and that the volatility is constant.

inflation process that satisfy, by construction, the property of Fisher neutrality.[9] Assumption (A3) relaxes the independence assumption in a flexible but technically tractable way.

We focus our analysis on an all-equity firm. In general, however, the presence of both corporate and personal taxes creates an incentive for firms to issue debt. DeAngelo and Masulis (1980) have shown that the corporate tax benefit of debt financing depends on the existence of nondebt tax shields. They present a model of optimal capital structure that incorporates the impact of debt and nondebt corporate tax shields. They argue that deductions for depreciation and tax-loss carryforwards are substitutes for the tax benefits of debt financing and lead to a market equilibrium in which each firm will have a unique interior optimum leverage decision. In such an equilibrium, positive leverage reduces the negative impact of inflation shocks.

**Assumption 4.** The real before-tax marginal productivity of capital depends on a vector of production factors described by the following stochastic differential equations:

$$\mathrm{d}y_t^i = z_t^i \mu_y^i \,\mathrm{d}t + \sigma_y^i \sqrt{z_t^i}\,\mathrm{d}W_t^{y_i} \quad 1 \leqslant i \leqslant k,$$

$$\mathrm{d}z_t^i = (\xi^i z_t^i + \zeta^i)\,\mathrm{d}t + \sigma_z^i \sqrt{z_t^i}\,\mathrm{d}W_t^{z_i} \quad 1 \leqslant i \leqslant k,$$

$$\mathrm{E}(\mathrm{d}W_t^{y^i}, \mathrm{d}W_t^{z^i}) = \rho_{y^i, z^i}\,\mathrm{d}t \quad 1 \leqslant i \leqslant k. \tag{A4}$$

The drift $\mu_y^i$ in Eq. (A4) makes the capital stock process non-stationary (consistent with the empirical evidence), while the state variables $z_t^i$ follow unconditionally stationary distributions. In the spirit of Longstaff and Schwartz (1992), this structure captures differential effects of the state variables $z_t^i$, for $i \leqslant k$, on the marginal productivity of capital. If $\mu_y^i \neq 0$ and $\sigma_y^i = 0$, the state variable $z_t^i$ affects only the instantaneous return on capital and not its local variance. If $\mu_y^j = 0$ and $\sigma_y^j > 0$, the state variable $z_t^j$ affects the local variance without changing the local mean of the return on capital. Thus, $\sigma_y^i$ is related to uncertainty (i.e. unexpected innovations) with respect to the marginal productivity of capital, while $\sigma_z^i$ characterizes the volatility of expected innovations in the marginal productivity of capital. In this sense, given estimates of the parameters $\mu_y^i$ and $\sigma_y^i$, we can relate the effects of pricing factor innovations to the dynamics of capital productivity. For the sake of generality, we allow the two Brownian motions $\mathrm{d}W_t^{y_i}$ and $\mathrm{d}W_t^{z_i}$ to be potentially correlated. Cox et al. (1985a) assume that the economic agent can form a portfolio of different production processes. It can be shown that in this economy the optimal portfolio composition is time invariant. Therefore, with no loss of generality, Eq. (A4) can be interpreted as the evolution equation of the optimal portfolio's marginal productivity. The dynamics are such that realized returns on physical

[9]Additional papers in which the inflation rate is considered as an independent and neutral process are, among others, Gibbons and Ramaswamy (1993), Pearson and Sun (1994), and Chen and Scott (1993).

investment $\mathrm{d}K_t/K_t$ are affected by the stochastic evolution of technological shocks $\mathrm{d}\mathbf{Y}_t$, the equilibrium price process $p_t^*$, and their covariance.

Nominal factors affect the real allocation of resources (and therefore equilibrium asset prices and risk premia) because of expected changes in price levels and inflation volatility. The expected change in price levels has been described. Eq. (A2) shows that the higher the volatility of inflation, the higher the volatility of the real capital accumulation process. A higher level of uncertainty with respect to future productivity decreases the optimal investment in real capital.

According to Cochrane and Piazzesi (2002), monetary shocks are important explanatory variables for bond excess returns. We directly explore their role in the structural model. Shocks to money demand and supply affect the price process through the monetary market-clearing condition. The monetary authorities have the following policy rule.

**Assumption 5.** The monetary authority sets the money supply $M_t^s$ on the basis of three nominal and real economic targets: (i) long-term target for nominal money growth equal to $-\theta_w/k_w$, (ii) an inflation target equal to $\bar{\pi}$, and (iii) an economic growth rate equal to $\bar{\kappa}$. Short-term deviations from the optimal long-run money growth are allowed to have level-dependent time-varying volatility. These properties are summarized as follows:

$$
\begin{aligned}
\frac{\mathrm{d}M_t^s}{M_t^s} &= w_t\,\mathrm{d}t + q_1\left(\frac{\mathrm{d}K_t^*}{K_t^*} - \bar{\kappa}\,\mathrm{d}t\right) + q_2\left(\frac{\mathrm{d}p_t^*}{p_t^*} - \bar{\pi}\,\mathrm{d}t\right) + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,\mathrm{d}W_t^M,\\
\mathrm{d}v_t &= (k_v v_t + \theta_v)\,\mathrm{d}t + \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}\,\mathrm{d}W_t^v,\\
\mathrm{d}w_t &= (k_\omega w_t + \theta_w)\,\mathrm{d}t + \sqrt{\sigma_{0w}^2 + \sigma_{1w}^2 w_t}\,\mathrm{d}W_t^w,\\
\mathrm{E}(\mathrm{d}W_t^v, \mathrm{d}W_t^M) &= \rho_{v,M}\,\mathrm{d}t, \quad \mathrm{E}(\mathrm{d}W_t^w, \mathrm{d}W_t^M) = 0.
\end{aligned}
\tag{A5}
$$

Since monetary authorities control the monetary aggregate only imperfectly through intermediate instruments, we model the evolution of the monetary aggregate as a stochastic process. The first factor, $w_t$, drives the conditional expected value of the money growth rate. The unconditional mean of $w_t$ is equal to $-\frac{\theta_w}{k_w}$. The second factor, $v_t$, drives the conditional volatility of the money growth rate.

When $q_1 = q_2 = 0$, monetary policy is exogenous. When $q_1 \neq 0$ (and/or $q_2 \neq 0$), real productivity (and/or inflation shocks) feeds back to the nominal side of the economy, not just because of the market-clearing condition for monetary holdings, but also because the central monetary authority reacts to deviations from the long-term economic growth and inflation targets $\bar{\kappa}$ and $\bar{\pi}$. The size of the parameters $q_1$ and $q_2$ quantify the intensity of the adjustments to the long-term real and nominal targets. When $\sigma_{1v}^2$ and $\sigma_{1M}^2$ are different from zero, the conditional volatilities of the monetary shocks are not constant. In this case, the risk premium on the nominal factor can be time varying, which, in principle, could account for the rejection of the expectations hypothesis so frequently found in the empirical literature.

The monetary policy rule is similar in spirit, but not isomorphic, to a Taylor (1993) rule. Taylor suggests that an effective way to approximate the optimal monetary policy is to increase short-term interest rates when short-term inflation expectations rise above the target level. In its pure form, money plays an active role only to the extent that it is helpful in predicting inflation. Several authors explore the welfare implications and robustness of a pure Taylor rule across a variety of models, relative to a classical money growth rate rule.[10] Christiano and Rostagno (2001) provide examples in which a pure Taylor rule can become a source of real economic instability. Their example echoes the message of Benhabib et al. (1999, 2001) who show that pathologies can occur even when the steady state associated with the inflation target of the Taylor rule is determinate and that monetary monitoring can be helpful.

Piazzesi (2004) studies the high-frequency empirical performance of an interest rate model in which the Fed policy rule targets the Fed funds rate. In her model, the target rate is assumed to follow a pure jump process with jump intensity dependent on the state of the economy and the Federal Open Market Committee calendar. The T-bill short rate is assumed to be the sum of the target rate and an exogenously specified stochastic spread following a square-root diffusion process. She posits an exogenous process for the stochastic discount factor and derives arbitrage-free bond prices. She finds that a Taylor rule that abstracts from interest rate smoothing predicts Fed-funds policy decisions accurately. This is consistent with our modeling approach. We use the money supply as a monetary instrument so that we close the model in general equilibrium using the market-clearing condition for monetary holdings. We obtain an endogenous price process without making exogenous assumptions about the spread between the Fed funds rate and the T-bill rate.

**Definition.** The representative agent equilibrium is defined as a vector $[C_t^*, M_t^{d*}, K_t^*, p_t^*]$ and a value function $J(K_t, z_t, v_t, w_t)$ that solve the following Hamilton-Bellman-Jacobi programming problem:

$$-\frac{\partial}{\partial t} J(K_t, \mathbf{z}_t, v_t, w_t) = \max_{\{c_t, M_t^d\}} \left\{ \mathscr{U}(X) + \mathscr{A} J(K_t, \mathbf{z}_t, v_t, w_t) \right\}$$

subject to the representative agent's utility (A1), the intertemporal budget constraint (A2), the taxation mechanism (A3), the factor structure (A4), the monetary policy (A5), and the money market-clearing condition $M_t^s = p_t^* M_t^{*d}$. We use $\mathscr{A}\phi$ to denote the differential operator applied to the function $\phi(X)$,

$$\mathscr{A}\phi(X) = \nabla_x \phi(X) \mu_x(X) + \tfrac{1}{2} Tr\{\nabla_{xx}\phi(X) \cdot \Sigma_x(X)\}$$

with $X$ being a multidimensional Ito process $\mathrm{d}X = \mu_x(X)\,\mathrm{d}t + \Sigma_x(X)\,\mathrm{d}W$.

To solve for equilibrium, we first solve for the optimal policy functions $\{C_t^*, M_t^{*d}\}$ of the representative agent, given $p_t^*$. These are obtained by solving the Hamilton-Bellman-Jacobi equation for a given price process $p_t$. We then solve for the

[10]See, among others, Ireland (2001), Rudenbush and Svensson (1999), Levin et al. (1999), Rotemberg and Woodford (1999), Gali et al. (2002), and Orphanides (2003).

equilibrium stochastic process of the general price level $p_t^*$ that satisfies the market-clearing condition for monetary holdings $M_t^s = p_t^* M_t^{*d}$, and is consistent with the budget constraint of the representative agent (A2) as well as monetary policy (A5). The proof is discussed in the Appendix. The main point of the derivation is that, in equilibrium, the process of both the general price level $p_t^*$ and the stock of capital $K_t^*$ are jointly determined.

In general, externalities and taxes break the connection between Pareto-optimal and competitive equilibrium allocations and are responsible for the failure of the Second Welfare Theorem. In these cases, solution methods for establishing the existence of a competitive equilibrium in taxed economies are model specific (for a discussion of these methods, see Stokey and Lucas, 1989, Chapter 18). However, when the tax rate on capital income is flat, subsidies take place via lump sum payments, individuals behave atomistically, and the technology has constant returns to scale. Tax distortions still affect the equilibrium allocation of resources and asset prices but they do not affect the aggregability conditions necessary for the Second Welfare Theorem to hold in this case, the tax distorted economy can be reparametrized into a pseudo-economy with a different marginal productivity of capital and discount rate such that the Second Welfare Theorem holds (Stokey and Lucas (1989, p. 547, example 18.2) highlight this result in the case of a discrete time economy).

**Proposition 1.** *The representative agent equilibrium is as follows*:

a. *The value function of the representative agent is*

$$\begin{aligned} J(t, K_t, \mathbf{z}_t, v_t, w_t) &= \mathrm{e}^{-\rho t} J(K_t, \mathbf{z}_t, v_t, w_t) \\ &= \frac{1}{\rho}\, \mathrm{e}^{-\rho t} \left[ P + Q \ln(\rho K_t) + \sum_{i=1}^{n} R_{z^i} z_t^i + R_v v_t + R_w w_t \right]. \end{aligned}$$

b. *The agent optimally allocates a constant fraction of wealth to consumption* $C_t^* = \frac{\rho}{Q} K_t$ *and to real monetary holdings* $M_t^{*d} = \gamma \frac{\rho}{Q} K_t = \gamma C_t^*$, *with* $Q = 1 + \gamma$.

c. *The equilibrium stochastic process of the real stock of capital is*

$$\frac{\mathrm{d}K_t^*}{K_t^*} = \mu_{K^*}(z_t^i, v_t, w_t)\,\mathrm{d}t + \sigma_{K^*}(z_t^i, v_t, w_t)\,\mathrm{d}W_t^{K^*}$$

*where the stochastic drift and variance are linear functions in the state variables* $\{z_t^i, v_t, w_t\}$:

$$\mu_{K^*}(z_t^i, v_t) = \mu_0^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) + \mu_{z^i}^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) z_t^i + \mu_v^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) v_t + \mu_w^{K^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) w_t$$

$$\begin{aligned} \sigma_{K^*}(z_t^i, v_t, w_t)\,\mathrm{d}W_t^{K^*} &= \left[ \frac{(1-q_2)}{(1-q_2) + \sigma_{y^i}(1-q_1)} \right] \\ &\quad \times \left[ -\tau_{\mathrm{cg}} \sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + \frac{\sigma_{y^i}}{(1-q_2)} \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,\mathrm{d}W_t^M \right] \end{aligned}$$

*The full specification of the drift parameters* $\mu_0^{K*}, \mu_{z^i}^{K*}, \mu_v^{K*}$, *and* $\mu_w^{K*}$ *is given in the appendix.*

d. *The equilibrium process for the endogenous general price level that clears the money market is*

$$\frac{\mathrm{d}p_t^*}{p_t^*} = \mu_{p*}(z_t^i, v_t, w_t)\,\mathrm{d}t + \sigma_{p*}(z_t^i, v_t, w_t)\,\mathrm{d}W_t^{p^*} \tag{1}$$

*where the stochastic drift and variance are linear functions of the state variables* $\{z_t^i, v_t\}$:

$$\mu_{p*}(z_t^i, v_t, w_t) = \mu_0^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi}) + \mu_{z^i}^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi})z_t^i + \mu_v^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi})v_t + \mu_w^{p^*}(\underline{q}, \underline{\tau}, \overline{k}, \overline{\pi})w_t$$

$$\begin{aligned}\sigma_{p*}(z_t^i, v_t, w_t)\,\mathrm{d}W_t^{p^*} = {} & \frac{(q_1-1)(1-q_2)\sigma_{y^i}}{(1-q_2)-\tau_{\mathrm{cg}}(1-q_1)}\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} \\ & - \frac{2\tau_{\mathrm{cg}}(q_1-1)+(1-q_2)}{(1-q_2)-\tau_{\mathrm{cg}}(1-q_1)}\sqrt{\sigma_{0M}^2+\sigma_{1M}^2 v_t}\,\mathrm{d}W_t^M.\end{aligned}$$

*The full specification of the drift parameters* $\mu_0^{p^*}, \mu_{z^i}^{p^*}, \mu_v^{p^*}$, *and* $\mu_w^{p^*}$ *is given in the appendix.*

The results of Proposition 1 show that if $\tau_{\mathrm{pr}}$ or $\tau_{\mathrm{cg}}$ are different from zero, nominal shocks have nonneutral implications on the real allocation of resources due to the lack of perfect indexation of the fiscal system. Moreover, when $q_1$ or $q_2$ are also different from zero, the monetary authority pursues an active economic policy that depends on the extent of deviations from real and nominal long-term economic targets.

Now we describe the equilibrium value of the term structure of interest rates. From the first-order conditions of the representative agent, the price of a nominal zero-coupon bond $B_t^\tau$ with time to maturity $\tau$ is equal to the conditional expected value of the intertemporal marginal rate of (consumption) substitution multiplied by the real payoff at maturity of the bond:

$$B_t^\tau = \mathrm{E}_t\left[\mathrm{e}^{-\rho\tau}\frac{\exp(-\ln X_{t+\tau}^*)\frac{1}{p_{t+\tau}^*}}{\exp(-\ln X_t^*)\frac{1}{p_t^*}}\right]. \tag{2}$$

Substituting the optimal consumption policy in (2) and defining, for convenience, $\kappa_t^* = (\gamma+1)\ln K_t^* + \rho t$, the equilibrium term structure of interest rates is the solution to the following stochastic problem:

$$B_t^\tau = \frac{1}{\exp(-\kappa_t^*)\frac{1}{p_t^*}}\mathrm{E}_t\left[\exp(-\kappa_{t+\tau}^*)\frac{1}{p_{t+\tau}^*}\right]. \tag{3}$$

Note that the vector diffusion process of $\{\kappa_t^*, p_t^*\}$ is non-stationary. However, in this case the Yamada-Watanabe Theorem guarantees the existence of a weak solution, unique in probability law for the stochastic process $\exp(-\kappa_t^*)\frac{1}{p_t^*}$. Moreover, since the growth conditions for the unbounded function $\exp(-\kappa_t^*)\frac{1}{p_t^*}$ are satisfied, we can apply a generalized version of the Feynman-Kac Theorem and obtain closed-form solutions for (3) by solving the dual representation of (3) expressed in differential form (a formal discussion of these conditions can be found in the appendix). Given equilibrium levels of the capital stock, the general price level, and the vector of state variables $(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*)$, if $B(\kappa_t^*, p_t^*, v_t^*, \mathbf{z}_t^*; \tau)$ is the solution to the stochastic problem (3), then it must also be the solution to the following differential problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} B(\kappa_t^*, p_t^*, v_t^*, w_t^*, \mathbf{z}_t^*; \tau) = \mathscr{A} B(\kappa_t^*, p_t^*, v_t^*, w_t^*, \mathbf{z}_t^*; \tau),$$

$$\text{s.t. } B(\kappa_t^*, p_t^*, v_t^*, w_t^*, \mathbf{z}_t^*; 0) = 1 \quad \forall t \text{ and } \tau.$$

The solution of the previous partial differential equation is summarized in the following proposition.

**Proposition 2** (*The nominal term structure*). *The nominal price of a nominal zero-coupon bond $B_t^\tau$ with time to maturity $\tau$ is a log-linear function of the real productivity and nominal shocks $z_t^i$, $v_t$ and $w_t$. The closed-form solution is*

$$B_t^\tau = A(\tau) \exp\left[-b_v(\tau)v_t - b_w(\tau)w_t - \sum_{i=1}^{n} b_{z^i}(\tau)z_t^i\right], \tag{4}$$

$$A(\tau) = \exp(A_0\tau)a_v(\tau)a_w(\tau)c_v(\tau)c_w(\tau)\prod_{i=1}^{n} a_{z^i}(\tau).$$

*Let us define $j = v,\ z^i, w$ so that*

$$b_j(\tau) = \frac{1}{2\Theta_2^j}\left[-\Theta_1^j + \sqrt{D^j}\,\tan(\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j})\right] \quad \text{where } D^j = -(\Theta_1^j)^2 + 4\Theta_0^j\Theta_2^j,$$

$$a_j(\tau) = 2^{A_j/\Theta_2^j}\exp(-\tfrac{A_j\tau\Theta_1^j}{2\Theta_2^j})\left[\cos(\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j})\right]^{A_j/\Theta_2^j}\left(\frac{\Theta_2^j\Theta_2^j}{D^j}\right)^{A_j/2\Theta_2^j},$$

$$\begin{aligned} c_j(\tau) = {} & \exp\left[\tfrac{1}{2\Theta_2^j}\left(B_j\Theta_2^j + B_j\tau\Theta_2^j - 2B_j\tau\Theta_2^j\Theta_2^j - B_v\Theta_1^j\log(1 + \tfrac{\Theta_2^j}{D^j})\right)\right] \\ & \times \exp\left[\tfrac{1}{2\Theta_2^j}\left(-B_j\sqrt{D^j}\,\tan[\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j}]\right)\right] \\ & \cdot\left[\cos\left(\arctan(\tfrac{\Theta_1^j}{\sqrt{D^j}}) - \tfrac{1}{2}\tau\sqrt{D^j}\right)\right]^{-B_j\Theta_1^j/(\Theta_2^j)^2}. \end{aligned}$$

$\{\Theta_0^{\cdot}, \Theta_1^{\cdot}, \Theta_2^{\cdot}, A_0, A_{\cdot}, B_{\cdot}\}$ *are functions of the structural parameters of the economy, the parameters controlling for the extent of nonneutrality of the fiscal system $\underline{\tau}$ and the monetary policy parameters $\underline{q}$. The proof of the proposition and the explicit functional forms of $\{\Theta_0^{\cdot}, \Theta_1^{\cdot}, \Theta_2^{\cdot}, A_0, A_{\cdot}, \underline{B}_{\cdot}\}$, in terms of the structural parameters, can be found in the appendix.*

The previous equilibrium pricing equation shows the following interesting features of the term structure for discount bond prices. The yield curve, defined as $-\frac{1}{\tau}\ln B_t^\tau$, is linear in the state variables. This property is induced by the linearity of the local variance in the pricing factors and is shared by models such as Cox et al. (1985b), Vasiček (1977), and others. Also, the long-term monetary targets $\overline{\pi}$ and $\overline{\kappa}$ affect the intercept of the yield curve but not the slope with respect to the factors. Next, the parameter $\underline{\tau}$, which describes the extent of nonneutrality, and the monetary policy parameters $\underline{q}$ affect both the intercept and the slope of the yield curve. Finally, the equilibrium pricing equation can accommodate a variety of yield curves. Each pricing factor has a different effect on the expected productivity of capital, through the stochastic drift, and its unexpected innovations. Depending on the values of the structural parameters, a positive shock to $z_t^i$ can vary the equilibrium risk premium of discount bonds by changing both the price and quantity of risk. This property will be important when explaining the violations of the expectations hypothesis of interest rates. Depending on the level of $z_t^i$, the yield curve can be monotonically increasing or decreasing. In addition, it can have a hump, a trough or both. This characteristic is shared by multifactor term structures models, such as Longstaff and Schwartz (1992) and Constantinides (1992), but not by simpler single-state variable models such as Vasiček (1977) and CIR (1985b). The instantaneous nominal interest rate can be computed from $r_t = -\frac{\mathrm{d}}{\mathrm{d}\tau}B_t^\tau|_{\tau=0} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\ln B_t^\tau|_{\tau=0}$ from which, using the boundary conditions on $B_t^\tau$ for $\tau = 0$, we obtain $r_t = -A_0 - \sum_i \Theta_0^{z^i} z_t^i - \Theta_0^v v_t - \Theta_0^w w_t$.

## 2.1. The price of risk

Duarte (2000), Dai and Singleton (2000), Backus et al. (1999), and Duffee (2002) show that when the market price of risk is assumed to be a fixed multiple of the risk level, traditional *completely* affine models fail to explain the conditional second moments of interest rates (Duffee uses the term "completely affine" to denote this specific class of affine models and provide intuition behind their inability to reproduce different shapes of the term structure or describe bond excess returns). For instance, completely affine models can neither duplicate the magnitude of the empirical violations of the expectations hypothesis nor generate excess returns with a small unconditional mean and high variance. Forecast errors are large and negatively correlated with the slope of the term structure. However, Duffee claims that "all is not lost" when the basic benchmark model is generalized to a specification in which the price of risk is no longer a constant multiple of interest rate volatility. His specification, labeled "essentially affine", adds substantial flexibility to the original class of affine models and allows the risk premium to become negative and delivers substantially better empirical properties. Our model offers a simple example of an economy supporting an equilibrium in which the endogenous price of risk is not a constant multiple of volatility. To see this, let $m_t$ be the stochastic discount factor, $m_t = \mathrm{e}^{-\rho t}U'(c_t^*, M_t^*)$, so that the discounted value of any tradable asset is a martingale, $B_t^\tau m_t = \mathrm{E}_t(m_{t+1}B_{t+1}^\tau)$. The diffusion process of the stochastic discount factor must be of the form

$$\frac{\mathrm{d}m_t}{m_t} = -r_t\,\mathrm{d}t - \Lambda_t\,\mathrm{d}\mathbf{W}_t$$

with $\Lambda_t\,\mathrm{d}W_t$ being the price of risk. From the equilibrium solution of the structural model, we obtain

$$\begin{aligned}
\mathbf{\Lambda}_t\,\mathrm{d}\mathbf{W}_t &= B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + B_{p*}\sigma_{p*}(\cdot)\,\mathrm{d}W_t^{p^*} \\
&= B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + \frac{B_p}{(1-q_2)+B_p(1-q_1)} \\
&\quad\times\left[(q_1-1)B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + \sqrt{\sigma_{0M}^2+\sigma_{1M}^2 v_t}\,\mathrm{d}W_t^M\right] \\
&= \left[B_z + \frac{B_p(q_1-1)B_z}{(1-q_2)+B_p(1-q_1)}\right]\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + \frac{B_p\sqrt{\sigma_{0M}^2+\sigma_{1M}^2 v_t}}{(1-q_2)+B_p(1-q_1)}\,\mathrm{d}W_t^M,
\end{aligned}$$

where $B_z$ and $B_p$ are known functions of the structural real and nominal parameters, respectively, as earlier defined.

*The expected risk premium*: From the bond price solution, we can obtain the following risk-neutral dynamics:

$$\begin{aligned}
\frac{\mathrm{d}B_t^\tau}{B_t^\tau} &= r_t\,\mathrm{d}t - b_v(\tau)\sqrt{\sigma_{0v}^2+\sigma_{1v}^2 v_t}\,\mathrm{d}\tilde{W}_t^v - b_\omega(\tau)\sqrt{\sigma_{0w}^2+\sigma_{1w}^2 w_t}\,\mathrm{d}\tilde{W}_t^\omega \\
&\quad - b_{z_i}(\tau)\sigma_z^i\sqrt{z_t^i}\,\mathrm{d}\tilde{W}_t^{z_i}.
\end{aligned}$$

From the correlation structure of the factors, with no loss of generality we can express $\mathrm{d}\tilde{W}_t^v$ in terms of two orthogonal components, one of which is correlated with $\mathrm{d}\tilde{W}_t^M$, i.e., $\mathrm{d}\tilde{W}_t^v = \rho_{M,v}\,\mathrm{d}\tilde{W}_t^M + \sqrt{1-\rho_{M,v}^2}\,\mathrm{d}\check{W}_t^v$, and similarly for $\mathrm{d}\tilde{W}_t^\omega$ and $\mathrm{d}\tilde{W}_t^{z_i}$, thus

$$\begin{aligned}
\frac{\mathrm{d}B_t^\tau}{B_t^\tau} &= r_t\,\mathrm{d}t - b_v(\tau)\sqrt{\sigma_{0v}^2+\sigma_{1v}^2 v_t}\left[\rho_{M,v}\,\mathrm{d}\tilde{W}_t^M + \sqrt{1-\rho_{M,v}^2}\,\mathrm{d}\check{W}_t^w\right] \\
&\quad - b_w(\tau)\sqrt{\sigma_{0w}^2+\sigma_{1w}^2 w_t}\left[\rho_{M,w}\,\mathrm{d}\tilde{W}_t^M + \sqrt{1-\rho_{M,w}^2}\,\mathrm{d}\check{W}_t^w\right] \\
&\quad - b_{z_i}(\tau)\sigma_z^i\sqrt{z_t^i}\left[\rho_{y^i,z^i}\,\mathrm{d}\tilde{W}_t^{y^i} + \sqrt{1-\rho_{y^i,z^i}^2}\,\mathrm{d}\check{W}_t^{z^i}\right].
\end{aligned}$$

To obtain the bond return process under the physical measure, we use the previous factor rotation and then Girsanov's theorem in which the change of drift is given by the price of risk: $\mathrm{d}W_t^M = -\Lambda_M\,\mathrm{d}t + \mathrm{d}\tilde{W}_t^M$ and $\mathrm{d}W_t^{y^i} = -\Lambda_{y^i}\,\mathrm{d}t + \mathrm{d}\tilde{W}_t^{y^i}$. Substituting the equilibrium price of risk, we find that expected bond excess returns are equal to

$$\mathrm{E}_t\left[\frac{\mathrm{d}B_t^\tau}{B_t^\tau} - r_t\,\mathrm{d}t\right] = b_v(\tau)\sqrt{\sigma_{0v}^2+\sigma_{1v}^2 v_t}\,\rho_{M,v}\Lambda_M + b_{z_i}(\tau)\sigma_z^i\sqrt{z_t^i}\,\rho_{y^i,z^i}\Lambda_{y^i}.$$

The ratio of expected bond excess returns and interest rate volatility is not constant but state-dependent:

$$\frac{\mathrm{E}_t\left[\frac{\mathrm{d}B_t^\tau}{B_t^\tau} - r_t\,\mathrm{d}t\right]}{Var[r_t]} = \phi(w_t).$$

Since the monetary factor $w_t$ drives the conditional expected value of the stochastic discount factor, it affects the level and volatilities of bond yields. However, since it is uncorrelated with unexpected innovations of the stochastic discount factor, it is not priced and does not affect bond expected risk premia. The market price of risk is not a constant multiple of the conditional interest rate volatility. Thus, depending on the level of $\omega_t$, bond prices can display high volatility without carrying a large expected risk premium, as discussed in Duffee (2002).

Moreover, when $\frac{\sigma_{0M}^2}{\sigma_{1M}^2} \neq \frac{\sigma_{0v}^2}{\sigma_{1v}^2}$ the market price of risk generated by monetary innovations, i.e., $\frac{B_p\sqrt{\sigma_{0M}^2+\sigma_{1M}^2 v_t}}{(1-q_2)+B_p(1-q_1)}$, is not proportional to the volatility of $\mathrm{d}v_t$, which is $\sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}$. The ratio between this component of the price of risk and the volatility of the monetary factor $v_t$ is state dependent and equal to

$$\frac{\Lambda_t(v_t)}{Vol(v_t)} = \frac{B_p}{(1-q_2)+B_p(1-q_1)}\left(\frac{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}\right)^{1/2}.$$

Although the price of risk does not change sign, if $-\frac{\sigma_{0M}^2}{\sigma_{1M}^2} > -\frac{\sigma_{0v}^2}{\sigma_{1v}^2}$ the equilibrium price of risk can become zero even for relatively high values of the volatility of the nominal factor.

### 2.2. *The real term structure and the inflation risk premium*

When the inflation rate affects the real capital accumulation process, the term structure of nominal bonds includes an inflation risk premium. The term structure of inflation risk premia is given by $Cov_t\left[\frac{\mathrm{e}^{-\rho\tau}U'(X_{t+\tau}^*)}{U'(X_t^*)};\frac{p_t^*}{p_{t+\tau}^*}\right]$ and the relation between the prices of nominal and index-linked bonds, $IL_t^\tau$, is given by

$$B_t^\tau = IL_t^\tau\,\mathrm{E}_t\left(\frac{p_t^*}{p_{t+\tau}^*}\right) + Cov_t\left[\frac{\mathrm{e}^{-\rho\tau}U'(X_{t+\tau}^*)}{U'(X_t^*)};\frac{p_t^*}{p_{t+\tau}^*}\right]$$

The closed-form solution of the term structures of index-linked bonds and of inflation risk premia are summarized in the following proposition.

**Proposition 3** (*The real term structure and the inflation risk premium*). (a) *The term structure of index-linked bonds and the expected value of the reciprocal of the equilibrium rate of inflation are both affine in the nominal* (*monetary*) *and real* *(productivity) factors $v_t$ and $z_t^i$:*

$$IL_t^\tau = A^{IL}(\tau)\,\exp\left\{-b_v^{IL}(\tau)v_t - b_w^{IL}(\tau)w_t - \sum_{i=1}^{n} b_{z^i}^{IL}(\tau)z_t^i\right\},$$

$$\mathrm{E}_t\left(\frac{p_t^*}{p_{t+\tau}^*}\right) = A^p(\tau)\,\exp\left\{-b_v^p(\tau)v_t - b_w^p(\tau)w_t - \sum_{i=1}^{n} b_{z^i}^p(\tau)z_t^i\right\}.$$

*The parameters $A^{IL}(\tau), b_v^{IL}(\tau), b_w^{IL}(\tau)$, and $b_{z^i}^{IL}(\tau)$ and $A^p(\tau), b_v^p(\tau), b_w^p(\tau)$, and $b_{z^i}^p(\tau)$ of the closed-form solution are functions of the structural parameters of the economy, the parameters controlling for the extent of nonneutrality of the fiscal system $\underline{\tau}$, and the monetary policy parameters $\underline{q}$. The explicit functional forms are provided in the appendix.*

(b) *From property* (a), *it follows that the term structure of inflation risk premia is not affine. Its solution is given by*

$$Cov_t\left[\frac{\mathrm{e}^{-\rho\tau}U'(X_{t+\tau}^*)}{U'(X_t^*)};\frac{p_t^*}{p_{t+\tau}^*}\right] = B_t^\tau - IL_t^\tau \cdot \mathrm{E}_t\left(\frac{p_t^*}{p_{t+\tau}^*}\right).$$

The solution for the inflation risk premium enables us to estimate the entire term structure of inflation risk premia, whose shape is sensitive to the extent of indexation to nominal shocks and to the monetary authority's response to deviations from the monetary targets, and also to study the empirical differences between short- and long-term inflation risk premia. We will explore these properties in Section ?. The inflation rate plays a nontrivial role in the closed-form solution of the nominal instantaneous interest rate. Since the inflation rate can distort the real capital accumulation process, nominal shocks can affect the nominal term structure by changing both the term structure of expected price levels and the real yield curve. Deviations from Fisher neutrality due to the level of the inflation rate and to changes in the volatility of inflation affect both the intercept and the slope of the spot rate schedule in different ways. The sensitivity of the spot interest rate with respect to $v_t$ is affected by the extent of heteroskedasticity of the inflation rate, namely $\sigma_{1v}^2$. The intercept of the instantaneous interest rate is affected only by $\sigma_{0v}^2$. Thus, derivative products that are more sensitive to changes in the slope of the term structure, such as constant maturity swaps and reverse floaters, are particularly sensitive to the extent of heteroskedasticity in monetary policy and productivity shocks.

## 3. Econometric methods

In this section, we use the restrictions obtained in Propositions 1 and 2 to estimate and test the overidentified representation of the economy using panel data on nominal bond yields. We then study the properties of the term structure of the inflation risk premium. We estimate the model by quasi maximum likelihood (QML) as in Chen and Scott (1993), Fisher and Gilles (1996), and Duffee (2002). The procedure assumes that the covariance matrix of the measurement errors is not of full rank, so that one can use a subset of bonds to reveal the unobservable risk factors by inverting the pricing equations. QML offers several advantages over alternative methods. Although QML does not use all the information implicit in the conditional density of the factors, it correctly takes into account both the first and second conditional moments. The literature usually finds a tradeoff in the ability of models to explain both. Moreover, Duffee and Stanton (2001) use Monte Carlo simulations to compare the small-sample properties of the efficient method of moments (EMM) with QML. They find that, despite its attractive asymptotic properties, the small-sample performance of EMM is worse than QML, especially in the case of the term structure of interest rates estimation which requires a large number of moment conditions. Based on their findings, they "advocate the use of QML methods when estimating dynamic term structure models, rather than using EMM at all" (specifically, they advocate the use of a linearized Kalman filter).

We present the three sets of first-moment conditions used in the QML estimation. These equilibrium moment conditions refer to the yields, the inflation rate, and the growth in monetary holdings, $\left[y_t^s, \ln \frac{p_{t+1}}{p_t}, \ln \frac{M_{t+1}^s}{M_t^s}\right]$. The closed-form solution for the expected equilibrium values are given by the functional forms $[\mathcal{M}^y, \mathcal{M}^p, \mathcal{M}^M]$. In what follows, we describe the specification of these three moment conditions.

- The bond pricing equation can be written as

$$y_t^\tau = \mathcal{M}^y(\underline{z}_t, v_t, w_t, \theta_o, \tau) + \eta_t^{y^\tau}, \quad \mathrm{E}\left[\eta_t^{y^\tau} | \underline{z}_t, v_t, w_t\right] = 0, \tag{5}$$

  where $y_t$ is the observed *bond-yield process* with time to maturity $\tau$, $\mathcal{M}(\cdot)$ is the moment restriction, $\theta_o \in \Theta$ is the vector of the structural parameters of the data-generating process, and $\eta_t$ is the $(n \times 1)$ vector of measurement errors. From Proposition 2, it follows that:

$$\begin{aligned}\mathcal{M}^y(\underline{z}_t, v_t, w_t, \theta_o) = & -\frac{1}{\tau}\ln A(\tau,\theta_0) + \frac{1}{\tau} b_v(\tau,\theta_0)v_t + \frac{1}{\tau} b_w(\tau,\theta_0)w_t \\ & + \frac{1}{\tau}\sum_i b_{z^i}(\tau,\theta_0)\, z_t^i.\end{aligned} \tag{6}$$

- The endogenous stochastic process of the price level depends on both monetary and productivity shocks. We split the log increments of the price process into two orthogonal components, $\mathcal{M}^{p^*}$ and $\eta^{p^*}$, namely the expected and unexpected inflation rates, conditional on $\{z_t^i, v_t, w_t\}$:

$$\frac{1}{T-t}\ln\frac{p_T^*}{p_t^*} = \mathcal{M}^{p^*}(\underline{z}_t, v_t, w_t, \theta^p) + \eta_t^{p^*}, \quad \mathrm{E}\left[\eta_t^{p^*} | \underline{z}_t, v_t, w_t\right] = 0. \tag{7}$$

  From Proposition 1, $(\mu^{p^*}, \sigma^{p^*})$ is a known function of the structural parameters. From Ito's Lemma, the instantaneous drift of the equilibrium log-price process is $\mu^{\ln p^*} \equiv \mu^{p^*} - \frac{1}{2}(\sigma^{p^*})^2$, which can be decomposed into the sum of four terms, $\mu^{\ln p^*} \equiv \mu_0^{\ln p^*} + \mu_{z^i}^{\ln p^*} z_t^i + \mu_v^{\ln p^*} v_t + \mu_w^{\ln p^*} w_t$. The solution of the conditional expected value at a one-month frequency is given by

$$\begin{aligned}\mathscr{M}^{p^*}(\underline{z}_t, v_t, w_t, \theta^p) = &\left[\mu_0^{\ln p^*} - \frac{\varsigma^i}{\xi^i}\mu_{z^i}^{\ln p^*} - \frac{\theta}{k}\mu_v^{\ln p^*}\right] \\ &+ \mu_{z^i}^{\ln p^*}\frac{(e^{\xi^i(T-t)}-1)}{\xi^i(T-t)}\left[z_t + \frac{\varsigma^i}{\xi^i}\right] \\ &+ \mu_v^{\ln p^*}\frac{(e^{k_v(T-t)}-1)}{k_v(T-t)}\left[v_t + \frac{\theta_v}{k_v}\right] \\ &+ \mu_w^{\ln p^*}\frac{(e^{k_w(T-t)}-1)}{k_w(T-t)}\left[w_t + \frac{\theta_w}{k_w}\right] \end{aligned} \tag{8}$$

which constitutes our second-moment restriction.

- The monetary authority follows an active monetary policy that is a function of both nominal and real economic targets. We express the equilibrium money supply as a function of expected and unexpected innovation, given $\{z_t^i, v_t, w_t\}$. The diffusion process for the money supply equation can be easily obtained by substituting the equilibrium values for the endogenous variables $\frac{dK^*}{k^*}$ and $\frac{dp^*}{p^*}$ in the policy function. Since the capital and price processes are affine (Proposition 1), the instantaneous drift $\mu^M$and volatility $\sigma^M$of the money supply process are affine in the nominal factors. Let us define $\mu^{\ln M^*} \equiv \mu^{M^*} - \frac{1}{2}(\sigma^{M^*})^2$, and decompose $\mu^{\ln M^*}$ into the sum of four terms, $\mu^{\ln M^*} \equiv \mu_0^{\ln M^*} + \mu_{z^i}^{\ln M^*} z_t^i + \mu_v^{\ln M^*} v_t + \mu_w^{\ln M^*} w_t$. We then solve for the conditional expected value of monetary holdings, which constitutes our third empirical moment restriction

$$\frac{1}{T-t}\ln\frac{M_T^s}{M_t^s} = \mathscr{M}^M(\underline{z}_t, v_t, w_t, \theta^M) + \eta_t^M, \quad \mathrm{E}\left[\eta_t^M | \underline{z}_t, v_t, w_t\right] = 0 \tag{9}$$

with

$$\begin{aligned}\mathscr{M}^M(\underline{z}_t, v_t, w_t, \theta^M) = &\left[\mu_0^{\ln M^*} - \frac{\varsigma^i}{\xi^i}\,\mu_{z^i}^{\ln M^*} - \frac{\theta}{k}\,\mu_v^{\ln M^*}\right] \\ &+ \mu_{z^i}^{\ln M^*}\frac{(e^{\xi^i(T-t)}-1)}{\xi^i(T-t)}\left[z_t + \frac{\varsigma^i}{\xi^i}\right] \\ &+ \mu_w^{\ln M^*}\frac{(e^{k_v(T-t)}-1)}{k_v(T-t)}\left[v_t + \frac{\theta_v}{k_v}\right] \\ &+ \mu_w^{\ln M^*}\frac{(e^{k_w(T-t)}-1)}{k_w(T-t)}\left[w_t + \frac{\theta_w}{k_w}\right]. \end{aligned} \tag{10}$$

QML also requires the computation of the second moments of $\left[y_t^\tau, \ln\frac{p_{t+\tau}^*}{p_t^*}, \ln\frac{M_{t+1}^*}{M_t^*}\right]$, the general equilibrium *solutions* of the model. The computation of the second moments would be standard if one knew the distribution of $v_t$. Unfortunately, the diffusion of the nominal factor is not a standard square-root process because the local volatility of $v_t$ is $\sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_t}$. The second moments can, however, be obtained in closed form using forward integration. The solution is summarized in the following Lemma.

**Lemma 1.** *The conditional second moments of the shifted square-root process $v_t$ are equal to*

$$\mathrm{E}_t(v_T^2) = v_t^2 \mathrm{e}^{2k(T-t)} + \left[\frac{1}{k}(\mathrm{e}^{2k(T-t)} - \mathrm{e}^{k(T-t)})\right](2\theta + \sigma_{1v}^2)\left(v_t + \frac{\theta}{k}\right) + \left[\frac{1}{2k}(\mathrm{e}^{2k(T-t)} - 1)\right]\left[\sigma_{0v}^2 - (2\theta + \sigma_{1v}^2)\frac{\theta}{k}\right]$$

*The conditional variance follows easily from* $Var_t(v_T) = \mathrm{E}_t(v_T^2) - \mathrm{E}_t^2(v_T)$.

## 4. The data

The empirical results are based on 492 monthly observations from January 1960 to December 2000. The dataset consists of three main components: interest rates, price levels, and money supply. Interest rate data from January 1960 to February 1991 are obtained from the McCulloch and Kwon dataset; see McCulloch (1990) and Kwon (1992). This database contains end-of-month zero-coupon yields and forward curves based on the McCulloch (1975, 1990) methodology from one month to ten years. We extend this dataset to 1998 by using the data provided by Duffee (2002)[11] and further to 2000 using the McCulloch methodology applied to original bond data.

Inflation data are based on the Consumer Price Index (CPI) for all urban consumers, which is available from January 1947. The money supply data are from the official H.6 release of the Federal Reserve Board of Governors. The data provided start from January 1959. We choose M2 as our measure of the money stock because it includes money market deposit accounts, which can be used for purchasing products and services, and is thus consistent with the definition of money in our model. For our purposes, M3 is too wide a measure since it includes instruments that pay significant interest rates and cannot therefore be classified as money in our framework.

The correlation between M2 growth and the yield on the five-year zero-coupon bond is 20%. Moreover, the monthly correlation between M2 growth and inflation is 16.8%. The relatively high correlation between money growth and both interest rates and inflation highlights the importance of explicitly considering the monetary side of the economy in order to explore the properties of the term structure of interest rates.

Fig. 1 illustrates the behavior of the spread between the one-year nominal interest rate and realized inflation. This spread is often used as a proxy for the real interest rate, which implicitly assumes that the inflation risk premium is zero and inflation follows a random walk. The spread declines in recessions and increases during economic expansions. Gray boxes on the graph show the periods of U.S. recessions compiled and reported by NBER. These are consistent with the dynamics of the real marginal productivity of capital over the business cycle. The only exception from this pattern is the 1981–1982 recession in which the real interest rate proxy remained at

[11]available at http://www.haas.berkeley.edu/~duffee.

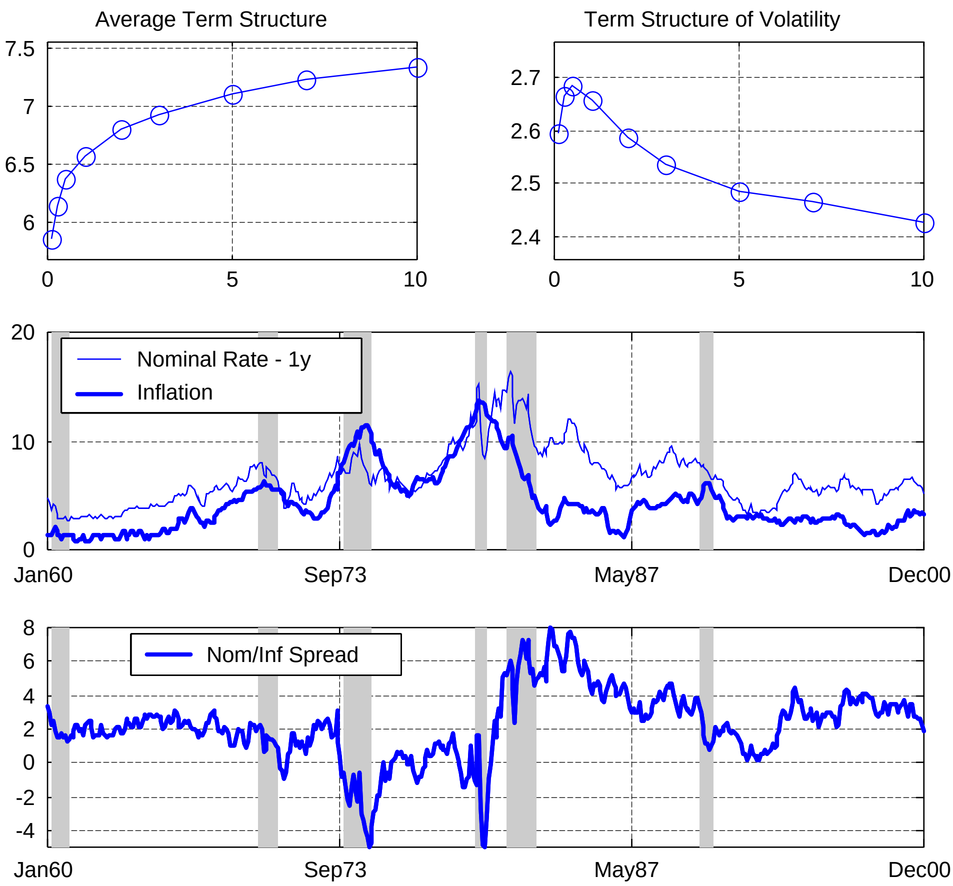


Fig. 1. The term structure over the sample. The figure summarizes the dataset. It is based on 444 monthly observations from January 1960 to December 2000. The top two pictures depict the sample mean of the level and volatility of the term structure of nominal interest rates. The central panel shows the evolution of the one-year nominal interest rate and CPI inflation rate. The bottom panel shows the evolution of the spread between the one-year nominal rate and inflation.

relatively high levels, perhaps due to an increase in the inflation risk premium during the recession.

## 5. Empirical results

Litterman and Scheinkman (1991) find that three factors explain most of the Treasury yield curve movements. Their empirical evidence has prompted most of the term structure literature to focus on the estimation of three-factor models of the yield curve. In addition to convention, we find that the choice of three factors can be justified by running specification tests based on the approach by Andrews and

Ploberger (1994) and Hansen (1996). These tests account for the parameters of the excluded factors to be unidentified under the null hypothesis. A Wald chi-square type test cannot be used to test the dimension of a factor model since its asymptotic distribution is, under the null hypothesis, degenerate. Hansen suggests a local alternative to the null hypothesis. We test a three- versus four-factor model and find that the three-factor specification (one real and two nominal factors) is not statistically worse than a four-factor specification. The results are significant at the conventional 5% confidence level.

We use yields on three-month, three-year, and ten-year zero-coupon bonds to invert the pricing restrictions and uncover the underlying factors. The remaining maturities in the estimation are one and six months and one, two, five, seven years.

The overall mean and median absolute errors of the term structure fitting are 13.95 and 10.52 basis points, respectively (Table 1, Panel A). This compares to 16 basis

Table 1
Goodness of fit by maturity
This table presents a goodness-of-fit summary for the model (Panel A) and a comparison of forecasting performance with Duffee (2002) essentially affine specifications (Panel B). The fitting errors are defined as the difference between the model-generated nominal spot rate and the observed nominal rate for the period January 1960 to December 2000. Panel B reports mean squared errors (RMSE) for month $t$ forecasts of month $t+i$ bond yields. The structural model is compared to the forecast errors obtained by Duffee (2002, Table VI) for his two best-performing completely and essentially affine three-factor specifications. To facilitate a direct comparison, like Duffe we measure yields in decimal form (i.e., 0.04 corresponds to 4%/year).

| Maturity | Mean absolute error | Median absolute error |
|---|---|---|
| *Panel A. average cross-sectional fitting errors* | | |
| 6 months | 13.7 bp | 9.3 bp |
| 1 year | 17.0 bp | 12.2 bp |
| 2 years | 8.5 bp | 6.3 bp |
| 5 years | 16.6 bp | 14.3 bp |
| 10 years | 12.3 bp | 11.6 bp |

| Bond maturity | Forecast horizon (months) | Completely affine Duffee (2002) | Essentially affine Duffee (2002) | Structural model |
|---|---|---|---|---|
| *Panel B. comparison of in-sample forecasting performance* | | | | |
| 6 months | 3 | 1.048 | 1.019 | 1.021 |
| 2 years | 3 | 0.883 | 0.853 | 0.867 |
| 10 years | 3 | 0.554 | 0.547 | 0.549 |
| 6 months | 6 | 1.427 | 1.368 | 1.375 |
| 2 years | 6 | 1.181 | 1.133 | 1.151 |
| 10 years | 6 | 0.772 | 0.764 | 0.765 |
| 6 months | 12 | 1.868 | 1.798 | 1.803 |
| 2 years | 12 | 1.583 | 1.544 | 1.545 |
| 10 years | 12 | 1.131 | 1.133 | 1.133 |

points in the completely affine three-factor CIR model estimated by Chen and Scott (1993, p. 25). This is remarkable since we fit not only the term structure but also the inflation rate and monetary holdings. Moreover, we fit the term structure up to a maturity of ten years, as opposed to five years as in Chen and Scott. The mean absolute error for the five year yield to maturity is 16.6 basis points. Since our model is not completely affine, we also compare its performance with the essentially affine specification studied by Duffee (2002) using a matching sample until 1994. In Table 1, Panel B, we summarize the in-sample forecasting performance of the model and compare it to the results in Duffee. We find that the forecasting errors are smaller than those implied by a three-factor completely affine model. Moreover, although our model is clearly less flexible than the three-factor affine specification advocated by Duffee, our forecasting errors are close to his.

### *5.1. Estimation and tests of overidentifying restrictions*

We test the overidentifying restrictions generated by the structural model with three factors. Under the null hypothesis that the model is correctly specified, $\varepsilon(\theta)'\Sigma^{-1}\varepsilon(\theta)$ is asymptotically $\chi^2$ chi-squared distributed. The optimal weighting matrix $\Sigma^{-1}$ is the inverse of the asymptotic covariance matrix of the residuals, estimated using a Newey-West estimator with 12 monthly lags. We find that the model with three factors has a $p$-value equal to 14%, suggesting that the overidentifying restrictions are not rejected. Changing the number of lags from zero to 12 yields a $p$-value ranging from 8% to 21%. We also find that the $p$-values are very robust as long as the three instruments used span the term structure, i.e., a short-term bond, a medium-term bond, and a long-term bond. The largest $p$-values are found when the three bonds used to invert the factors are the three-month, three-year, and ten-year bonds. The result is robust to the number of lags used in the Newey-West covariance matrix.

Four main reasons account for the better empirical performance of this model. First, the capital evolution process is more general than in a standard CIR model since it includes stochastic innovations for both the level and uncertainty in the drift of the marginal productivity of capital. This gives an important element of flexibility to the model. Second, we model the nominal side of the economy using shifted square-root processes. Their conditional distribution is different from the original CIR square-root processes and their realizations can be negative. Third, the tax implications of the nominal side of the economy have a crucial effect on the dynamics of the nominal term structure. Fourth, and perhaps most important, the price of risk is not a constant multiple of volatility.

Estimates of the model's parameters and their corresponding standard errors are presented in Table 2. The asymptotic covariance matrix of the parameters is based on the outer product of the Jacobian of the log-likelihood function. The parameters $\tau_{pr}$ and $\tau_{cg}$, the income and capital gains tax, capture the fiscal side of the economy. The estimated values of these parameters are 27% and 29%, respectively, statistically significant at the 1% confidence level. The capital gains tax estimate is very close to the actual historical value of the tax rate; the income tax rate estimate, however, is

Table 2

Parameter estimates based on maximum likelihood estimation

This table presents the maximum likelihood parameter estimates and their standard errors. The estimation is based on 492 monthly observations from January 1960 to December 2000. The estimation is based on nine maturities ranging from one month to ten years. The estimated model has three factors. The two real factors follow $dz_t^i = (\xi^i z_t^i + \zeta^i)\,dt + \sigma_z^i \sqrt{z_t^i}\,dW_t^{z_i}$ where $i = 1, 2$. The capital depreciation rate is denoted by $\lambda_m$; the capital setup cost is $\lambda_s$; and the capital gains and profits tax are $\tau_c$ and $\tau_p$, respectively. The Brownian motions $W_t^{y_i}$ and $W_t^{z_i}$ are correlated with correlation coefficient $\rho_{y_i z_i}$. The monetary authority sets the money supply $M_t^s$ endogenously as $dM_t^s/M_t^s = w_t\,dt + q_1(\frac{dK_t^*}{K_t^*} - \bar{\kappa}\,dt) + q_2(\frac{dp_t^*}{p_t^*} - \bar{\pi}\,dt) + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,dW_t^M$. Moreover, $v_t$ and $w_t$ follow shifted square root processes. In parentheses we report the asymptotic standard deviations. In order to identify the depreciation parameter $\lambda_m$, we set $\ell$ to the time-series average of the *public Transfers/gnp* ratio.

| | | | | | |
|---|---|---|---|---|---|
| $\lambda_m$ | $\lambda_s$ | $\tau_{pr}$ | $\tau_{cg}$ | $\gamma$ | |
| 0.14 | 0.05 | 0.27 | 0.29 | 0.43 | |
| (0.06) | (0.01) | (0.08) | (0.08) | (0.16) | |
| $q_1$ | $q_2$ | $\bar{\kappa}$ | $\bar{\pi}$ | $\rho$ | |
| 1.132 | −1.068 | 0.034 | 0.043 | 0.06 | |
| (0.37) | (0.35) | (0.01) | (0.01) | (0.04) | |
| $\sigma_{M_0}$ | $\sigma_{M_1}$ | $k_v$ | $\theta_v$ | $\sigma_{v1}$ | $\sigma_{v0}$ |
| 2.12 | 1.44 | −0.154 | 0.014 | 0.064 | 0.144 |
| (0.90) | (0.50) | (0.06) | (0.004) | (0.02) | (0.04) |
| $k_w$ | $\theta_w$ | $\sigma_{0w}$ | $\sigma_{1w}$ | $\rho_{M,v}$ | |
| −0.17 | 0.012 | 0.056 | 0.032 | 0.20 | |
| (0.06) | (0.05) | (0.02) | (0.01) | (0.08) | |
| $\zeta_1$ | $\xi_1$ | $\sigma_{z_1}$ | $\mu_{y_1}$ | $\sigma_{y_1}$ | $\rho_{z_1 y_1}$ |
| 0.011 | −0.44 | 0.03 | 0.576 | 0.56 | −0.61 |
| (0.004) | (0.15) | (0.01) | (0.21) | (0.20) | (0.23) |

substantially smaller than the statutory income tax rate, though very close to the effective income tax rate computed by Graham (1996). He considers a sample of 71,311 observations spanning 1980–1991 and computes the effective income tax rate taking into account the effects of tax-loss carryforwards and carrybacks, the investment tax credit, and the alternative minimum tax. He finds that the effective tax rate is 26.9% under the assumption of perfect foresight of future cash flows. The Institute on Taxation and Economic Policy carries over a similar study for the 1990s and finds an average effective tax rate of 26.5% (McIntyre and Coo Nguyen, 2000). These values are not statistically different from our estimated effective tax rate. The statutory long-term capital gains tax is currently 20%. However, over the period 1987–1996 and before 1980, it was 28%.

In order to identify the depreciation parameter $\lambda_m$, we exogenously set $\ell$ to be equal to the time-series average of the Transfer Payment to Fixed Assets (Durables and Services) ratio. The data are obtained from the National Income and Product Accounts of the United States. Given this identifying restriction, the gross depreciation rate $\lambda_m$ is 0.14, with a standard deviation of 0.06. The parameter $\lambda_s$ captures the variable-cost component of investment and is equal to 5%. It is significantly different from zero at the 1% confidence level. Estimates of $\theta_w$ and $k_w$ can be used to calculate the expected long-term growth of the money supply, which equals the long-run expected value of the nominal factor $-\theta_w/k_w$, or about 7%. The long-run inflation target $\overline{\pi}$ of the monetary authority is about 4.3%. The long-run real growth target $\bar{\kappa}$ is 3.4%.

## 6. The inflation risk premium

### 6.1. Cross-sectional properties

The average term structure of the inflation risk premium, calculated over the entire sample, is illustrated in Fig. 2, Panel D. The term structure is upward sloping. At a three-month horizon the inflation risk premium is 25 basis points, increasing to 70 basis points at a ten-year horizon.

The positive slope is due to two effects. First, it is usually more difficult to predict inflation at longer horizons. In the short term, the inflation rate behaves quite closely to a random walk. In the long term, the inflation rate is determined by monetary policy and, in a democratic society, by public support for achieving inflation targets. Public support is difficult to forecast and is influenced by a broad range of factors, including demographics, the level of the fiscal deficit, and international shocks. Thus, the higher uncertainty on long-term inflation rates translate into a positive slope for the inflation risk premium term structure. Second, the same inflation forecast errors have larger costs to investors in long-term bonds than to investors in short-term bonds. The greater duration of long-term bonds amplifies the dollar impact of the same amount of inflation uncertainty, inducing long-term bonds to carry a larger inflation risk premium.

The volatility term structure of the inflation risk premium is monotonically increasing,[12] from 4.5 basis points at a three-month horizon to 31 basis points at a ten-year horizon. We also compute the volatility of the percentage deviations from the mean in order to control for the different size of the premium at different horizons. The volatility curve of the inflation risk premium is still sharply upward sloping ranging from 18% (three months) to 48% (ten years). This result contrasts with the overall (hump-shaped) downward-sloping volatility yield curve.

[12]Detailed tables and figures are available from the authors upon request.

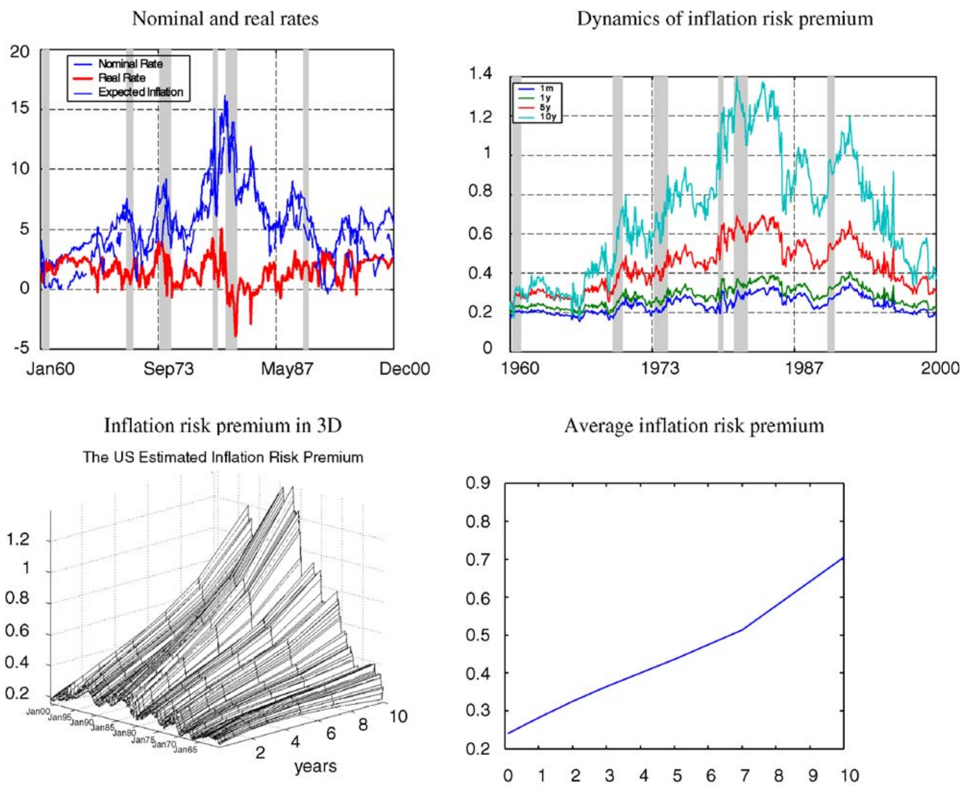


Fig. 2. The inflation risk premium. Panel A plots the realized nominal and real interest rates and the expected inflation rate. Panel B shows the time variation of the inflation risk premium for different time horizons. U.S. recession periods are marked as gray boxes. Panel C shows the three-dimensional evolution of the term structure of inflation risk premia. Panel D shows the average term structure of inflation risk premia over the entire sample.

### 6.2. Time-series properties

Is the conditional inflation risk premium time varying? Fig. 2, Panel B, illustrates the business cycle evolution of the inflation risk premia. Fig. 2, Panel C, illustrates the evolution of the total inflation risk premium in both the time and maturity domains. At a ten-year horizon, the inflation risk premium has fluctuated between 0.20% and 1.40%. This feature arises both from the time-varying conditional volatility of the risk factors and from the time variation of the price of risk. The nominal risk factors follow shifted square-root processes. If $\sigma_{1v}$ were equal to zero, then $v_t$ would not induce time variation in the inflation risk premium. We test the null hypothesis $\mathrm{H}_0 : \sigma_{1v} = 0$, and reject it. This suggests that the inflation risk premium in the nominal term structure is time varying and that an important component of its time variation is the monetary factor.

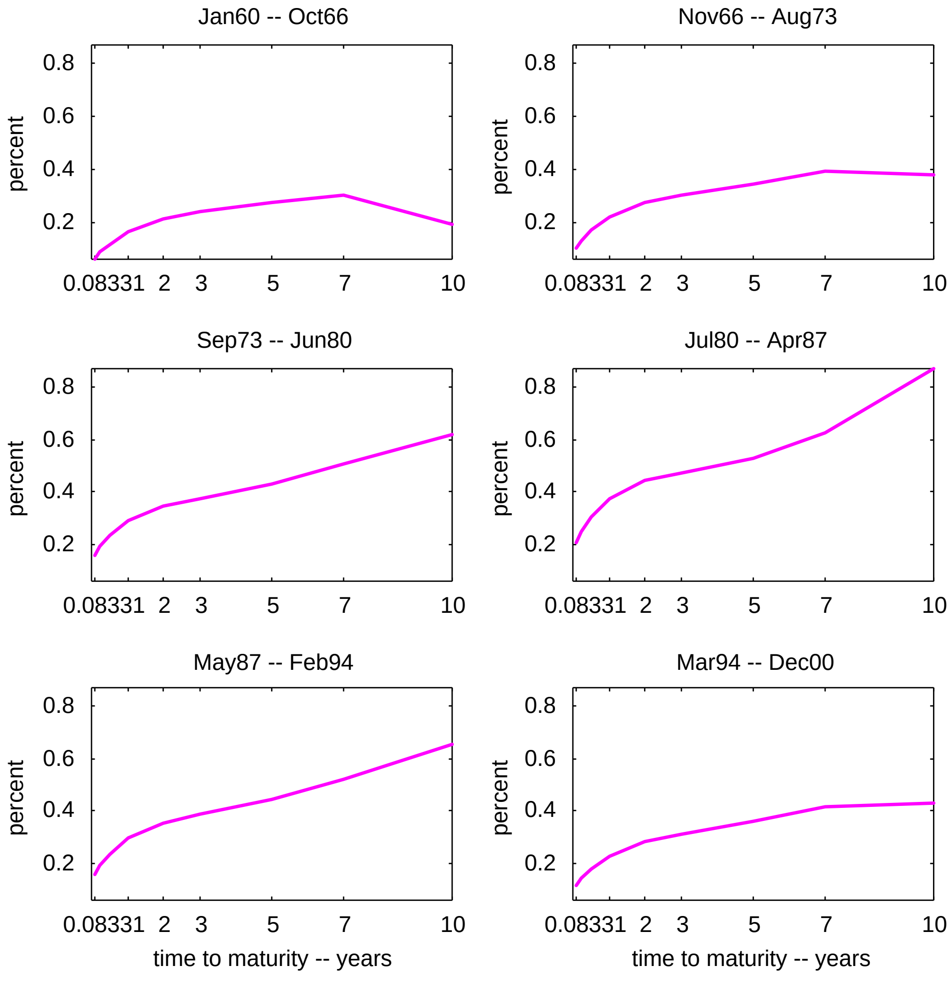


Fig. 3. Term structure of inflation risk premium by subperiods. This figure presents the variation of the average term structure of the inflation risk premium in six subperiods of equal length.

Fig. 2 shows that the inflation risk premium steadily rise from the early 1960s throughout the 1960s and 1970s to reach a peak in the 1979–1983 period. During this time the Fed changed the monetary target twice, once in 1979 and again in 1982,[13] and the level and volatility of inflation reached their highest post-war values. The inflation risk premium is the compensation required by the marginal investor to hold a nominal asset and bear the associated inflation risk. Thus, during this period the high inflation risk premium suggests a lack of confidence by the market in the monetary authorities' effectiveness in pursuing an inflation target. This ex ante lack

[13]In 1979 the Fed changed its target from the federal funds rate to money growth and nonborrowed reserves. In 1982 the Fed decided to target borrowed reserves instead.

of credibility, which is reflected in the time-series behavior of the level and slope of the inflation risk premium term structure, is confirmed ex post by a very high realized inflation rate volatility.

We split the overall sample into six equal subperiods (of six years and ten months each) to assess the effect of different regimes on the characteristics of the inflation risk premium. Fig. 3 illustrates the results. In the subperiod July 1980 to April 1987, the slope of the inflation risk premium is substantially higher than in any other subperiod.

An inspection of Fig. 2 shows that during periods of high nominal interest rates and inflation, such as the 1982 recession, the drop in real interest rates is correlated with a substantial increase in the inflation risk premium. The correlation between U.S. economic growth, proxied by the GDP growth rate, and the value of the inflation risk premium varies from −0.27 to −0.22 depending on the horizon. This highlights the fallacy in extrapolating the expected inflation rate by subtracting a constant inflation risk premium from the spread between nominal and index-linked bond yields. This is useful for capital budgeting, as the inflation risk adjustment for the cost of capital shows substantial time variation and is sensitive to the inflation regime.

Table 3 shows that expected inflation, nominal rates, and the inflation risk premium are positively correlated. The correlation between real rates and the inflation risk premium is substantially smaller. Since low levels of the real rate are

Table 3
Correlation matrix for model generated series
This table presents the correlation matrix for the estimated time series using the structural model. The first three variables are nominal interest rates at maturities of one month, five years, and ten years. The following three variables are real interest rates for the same maturities. Rp1m, Rp5y, and Rp10y indicate the inflation risk premium at a one-month, five-year, and ten-year horizon, respectively. The last variable, ModInf, is the expected inflation rate estimated with the structural model.

| | N3m | N5y | N10y | R3m | R5y | R10y | Rp3m | Rp5y | Rp10y | ModInf |
|---|---|---|---|---|---|---|---|---|---|---|
| N3m[a] | 1.00 | 0.89 | 0.83 | −0.44 | −0.50 | −0.28 | 0.12 | 0.49 | 0.55 | 0.72 |
| N5y | | 1.00 | 0.99 | −0.71 | −0.42 | −0.10 | 0.48 | 0.75 | 0.80 | 0.64 |
| N10y | | | 1.00 | −0.72 | −0.37 | −0.03 | 0.62 | 0.85 | 0.89 | 0.61 |
| R3m[b] | | | | 1.00 | 0.68 | 0.42 | −0.51 | −0.59 | −0.62 | −0.50 |
| R5y | | | | | 1.00 | 0.93 | 0.05 | −0.15 | −0.19 | −0.65 |
| R10y | | | | | | 1.00 | 0.36 | 0.20 | 0.17 | −0.49 |
| Rp3m[c] | | | | | | | 1.00 | 0.92 | 0.89 | 0.13 |
| Rp5y | | | | | | | | 1.00 | 1.00 | 0.40 |
| Rp10y | | | | | | | | | 1.00 | 0.44 |
| ModInf[d] | | | | | | | | | | 1.00 |

[a]N3m, N5y, and N10y are the nominal yields at a three-month, five-, and ten-year horizon, respectively.
[b]R3m, R5y, and R10y are the real yields at a three-month, five-, and ten-year horizon, respectively.
[c]Rp3m, Rp5y, and Rp10y indicate the inflation risk premium at a three-month, five-, and ten-year, respectively.
[d]ModInf is the expected inflation rate estimated with the structural model.

usually a characteristic of recessionary periods, we find that the inflation risk premium is on average higher in recessions.

There is little empirical evidence on the size of the U.S. inflation risk premium with which to compare our results. Most existing evidence focuses on international data to take advantage of index-linked bonds data in the U.K. (Evans, 2003; Shen, 2001). The history of traded index-linked bonds in the U.S. is very short, as they were issued for the first time on January 29, 1997. Moreover, until recently it has been quite an illiquid market. Shen (2001) uses 1996–1997 U.K. data on the spread between nominal gilts and index-linked bonds to estimate the inflation risk premium by subtracting the survey expected inflation from the (nominal minus index-linked) yield spread. He finds the average ten year inflation risk premium to be 74 basis points in the 1996–1997 period. For the same period, we find the U.S. ten year inflation risk premium to be about 60 basis points (70 basis points in the last 40 years). This is plausible since the volatility of inflation, and thus the economic cost of uncertainty about its future value, was lower in the U.S. during this time.

Ang and Bekaert (2004) use a regime-switching affine model of the term structure to estimate the U.S. nominal and real yield curves. In their model, the absolute size of the long-term implied inflation risk premium is 97 basis points, which is roughly one standard deviation higher than both our estimate (70 basis points) for the U.S. and Shen's estimate for the U.K. Although the difference is less than 30 basis points, their estimates seem high in light of the average bond-TIPS yield spread and the fact that in the last 40 years U.S. inflation rate volatility has been 50% of the U.K.'s figure (the mean and standard deviation of the U.S. inflation rate are 3.80% and 3.03%, compared to 4.88% and 6.03%, respectively, in the U.K.).

When a firm pays its cash flows to equity through share repurchases, and individuals can optimally defer their capital gains, there is an opportunity for tax savings. The value of the option to defer depends on the timing of the firm's cash flow distributions, the way those distributions are split between dividends and share repurchases, and the volatility of cash flows. Green and Hollifield (2003) simulate an all-equity firm in which capital distribution occurs solely via share repurchases. They estimate the tax-timing option under several different scenarios. When the volatility of cash flows is 20%, the tax-timing option reduces the present value of taxes paid by 8.18% (with respect to the taxes payable under full taxation, see Green and Hollifield, 2003, Table 3). This reduction is smaller if some of the profits are also distributed via dividends, and larger for a lower cash flow volatility. Since the inflation risk premium is proportional to the tax liability, our basis point estimate of the risk premium should be overstated by about the same percentage amount. This could imply a 4.8 basis point bias in the inflation risk premium, which is relatively small with respect to the estimated level of the inflation risk premium.

### 6.3. Decomposition of the inflation risk premium

We decompose the inflation risk premium volatility into two components: the first due to the time variation of the monetary factors, and the second due to the real factor. We study the relative importance of these two factors in driving the time

Table 4
Decomposition of inflation risk premium volatility
This table summarizes the results of a decomposition of the inflation risk premium volatility into nominal ($w_t$ and $v_t$) and real components ($z_t$). Let $\Delta_z$, $\Delta_w$, and $\Delta_v$ be the partial derivatives of the inflation risk premium $IRP_t$ with respect to the state variables $z_t$, $w_t$, and $v_t$, so that

$$Var(IRP_t) = \Delta_z^2\, Var(z_t) + \Delta_w^2\, Var(w_t) + \Delta_v^2\, Var(v_t).$$

| Horizon | Contribution to the time variation | | | |
|---|---|---|---|---|
| | Full sample | | Excluding 1979–1982 | |
| | Monetary (%) | Real (%) | Monetary (%) | Real (%) |
| 3 months | 62 | 38 | 60 | 40 |
| 6 months | 62 | 38 | 59 | 41 |
| 1 year | 61 | 39 | 57 | 43 |
| 2 years | 60 | 40 | 54 | 46 |
| 5 years | 58 | 42 | 51 | 49 |
| 10 years | 57 | 43 | 50 | 50 |

variation of the inflation risk premium. Let $\Delta_z(\hat{\Theta})$, $\Delta_w(\hat{\Theta})$, and $\Delta_v(\hat{\Theta})$ be the partial derivatives of the inflation risk premium $IRP_t$ with respect to the state variables $z_t$, $w_t$, and $v_t$. Then

$$Var(IRP_t) = \Delta_z^2(\hat{\Theta})\, Var(z_t) + \Delta_w^2(\hat{\Theta})\, Var(w_t) + \Delta_v^2(\hat{\Theta})\, Var(v_t).$$

Table 4 summarizes the relative contributions of the two factors to the total time variation of the inflation risk premium. We find that at a one-year horizon, 61% of the inflation risk premium volatility is due to the monetary factor. This highlights the extent to which the inflation risk premium is a monetary phenomenon. The result is surprisingly robust when we consider a subset that excludes the 1979–1982 period. We also find that the contribution of monetary shocks is negatively related to the horizon of the risk premium. At a ten-year horizon, the contribution of the monetary factor is 57% of the total volatility of the inflation risk premium, showing that the bond market anticipates that monetary shocks have a persistent effect.

## 7. The expectations hypothesis

One of the most debated and studied financial relation is the expectations hypothesis, hereafter EH, of interest rates. If the EH were correct, at least in a statistical sense, one could use implied forward rates to obtain a good proxy for the expected future spot rate. Unfortunately, most of the empirical evidence rejects the EH hypothesis. Such empirical evidence is important since it suggests the existence of a time-varying risk premium and since the direction and magnitude of rejection can be used as a direct metric to test and improve the specification of asset pricing models. Such a metric is directly related to the properties of the conditional second moments of interest rates.[14]

In what follows, we explore the extent to which our structural model can explain empirical rejections of the EH. We also explore whether rejection of the EH is due to a time-varying *inflation* risk premium or to a time-varying *real* (technological) risk premium. We can address this question by using the structural model to separately identify the two sources of shocks.

Dai and Singleton (2000) show that completely affine models cannot reproduce the magnitude of empirical deviations from the EH. A problem with this class of affine models is their assumption that the market risk premium is proportional to the local volatility of the factors. Duffee (2002) discusses the importance of this assumption and its more general implications for the term structure of interest rates. Both articles suggest reduced-form models that overcome this limitation without losing tractability. As discussed in Section 2.1, the (endogenous) price of risk in our model is not a constant multiple of local volatility. We explore, therefore, whether this generalization can provide a structural explanation for deviations from the EH.

Traditional tests of the EH, such as Campbell and Shiller (1991) and Dai and Singleton (2000), are typically based on the unbiased EH (U-EH). This version states that forward rates are unbiased predictors of futures rates once adjusted for a constant term premium. Although Cox et al. (1981) claim that this specification is incompatible with any continuous-time rational-expectation economy, McCulloch (1993) shows an example of a homoskedastic stochastic endowment economy that supports bond prices satisfying the U-EH in its pure form (with no term premium).

We therefore compute the forward premium for the monetary equilibrium and discuss the conditions under which the U-EH holds. Let the forward interest rate at time $t$ for an instantaneous forward contract beginning at time $T = t + \tau$ be $f(t, T)$. The instantaneous forward rate is equal to $-\frac{\partial}{\partial T}\ln B_t^\tau$, so that

$$f(t,T) = -\frac{A'(\tau)}{A(\tau)} + b'_v(\tau)v_t + b'_w(\tau)w_t + \sum_i b'_{z^i}(\tau)z^i_t. \tag{11}$$

Based on this, the following proposition characterizes deviations from the U-EH.

**Proposition 4** (*The unbiased expectations hypothesis*). *The matching-maturity forward rate is a conditionally biased estimator of the expected future spot interest rate. The forward premium is time varying and depends linearly on the level of the underlying*

[14]The Campbell and Shiller tests of the EH focus on the properties of the slope coefficient of a regression of future yield changes onto the current slope of the term structure. Since this slope coefficient is a ratio between a conditional covariance and a conditional variance, the ability of a model to reproduce the empirical violations of the EH are a function of the ability of the model to reproduce the empirical conditional second moments of interest rates.

*pricing factors as follows*:

$$
\begin{aligned}
f(t,T) - \mathrm{E}_t[r_T] = \Bigg[ & A_0 - \frac{A'(\tau)}{A(\tau)} + \frac{1}{2}\frac{(\Theta_1^v)^2 + 2\Theta_0^v\Theta_2^v}{\Theta_2^v}\frac{\theta_v}{k_v}(\mathrm{e}^{-k_v\tau} - 1) \\
& + \frac{1}{2}\frac{(\Theta_1^w)^2 + 2\Theta_0^w\Theta_2^w}{\Theta_2^w}\frac{\theta_w}{k_w}(\mathrm{e}^{-k_w\tau} - 1) \\
& + \frac{1}{2}\sum_{i=1}^{n}\frac{(\Theta_1^{z^i})^2 + 2\Theta_0^{z^i}\Theta_2^{z^i}}{\Theta_2^{z^i}}\frac{\zeta^i}{\xi^i}(\mathrm{e}^{-\xi^i\tau} - 1)\Bigg] \\
& + [b_v'(\tau) + \Theta_0^v\mathrm{e}^{-k_v\tau}]v_t + [b_w'(\tau) + \Theta_0^w\mathrm{e}^{-k_w\tau}]w_t \\
& + \sum_{i=1}^{n}[b_{z^i}'(\tau) + \Theta_0^{z^i}\mathrm{e}^{-\xi^i\tau}]z_t^i
\end{aligned}
\tag{12}
$$

The sign of the term premium can be strictly positive or negative, depending on the values assumed by the state variables. This depends on the stochastic volatility structure of the pricing factors, which makes the term premium time varying. Moreover, the presence of multiple factors allows different possible shapes of the term premium.

We now ask the following question: If we generate term structure data using the structural model and run Campbell-Shiller (1991) regressions, do we find the same pattern in the slope coefficients? If we do, the model would be able to capture the empirical properties of the conditional second moments, describe the characteristics of time variation of the forward risk premium, and link EH rejection to a structural economic explanation.[15]

Campbell and Shiller (1991) regress the change in the constant time-of-maturity yield onto the current slope of the yield curve. Let $y_t^\tau = -\frac{1}{\tau}\ln B(t,\tau)$ be the yield on a Treasury bond with maturity $t+\tau$:

$$
y_{t+m}^{n-m} - y_t^n = \alpha + \beta_{n,m}\underbrace{\left[\left(\frac{m}{n-m}\right)\left(y_t^n - y_t^m\right)\right]}_{S_t^{n,m}} + \varepsilon_t\,.
$$

The expectations hypothesis requires that $\beta_{n,m} = 1$. Since in our structural model the yield curve is affine in the states, we can solve for the regression coefficients in closed form.

**Proposition 5.** *The linear regression coefficient* $\beta_{n,m}$ *implied by the structural model is equal to*

$$
\beta_{n,m} = \frac{(n-m)}{m\phi(n,m)}\left\{\left[\frac{b_v(n-m)}{n-m}b_v^0(n,m)\right]\left(\frac{\sigma_{1v}^2\theta_v}{2k_v^2} - \frac{\sigma_{0v}^2}{2k_v}\right)\mathrm{e}^{km}\right.
$$

[15] In a reduced form setting, a similar question is explored by Dai and Singleton (2000).

$$
\begin{aligned}
& -\left[\frac{b_v(n)}{n} b_v^0(n,m)\right]\left(\frac{\sigma_{1v}^2\theta_v}{2k_v^2}-\frac{\sigma_{0v}^2}{2k_v}\right) \\
& \times\left[\frac{b_w(n-m)}{n-m} b_w^0(n,m)\right]\left(\frac{\sigma_{1w}^2\theta_w}{2k_w^2}-\frac{\sigma_{0w}^2}{2k_w}\right)\mathrm{e}^{k_w m} \\
& -\left[\frac{b_w(n)}{n} b_w^0(n,m)\right]\left(\frac{\sigma_{1w}^2\theta_w}{2k_w^2}-\frac{\sigma_{0w}^2}{2k_w}\right) \\
& +\sum_{i=1}^{n}\left[\frac{b_{z^i}(n-m)}{n-m} b_{z^i}^0(n,m)\right]\frac{\sigma_z^2\zeta}{2\xi^2}\mathrm{e}^{\xi m}-\sum_{i=1}^{n}\left[\frac{b_{z^i}(n)}{n} b_{z^i}^0(n,m)\right]\frac{\sigma_z^2\zeta}{2\xi^2}\Bigg\},
\end{aligned}
$$

*where $\phi(n,m)$ and $b^0(n,m)$ are known functions of the structural parameters.*

Let $\beta(\hat{\Theta})$ be the slope coefficient of the Campbell-Shiller regressions implied by the model for a set of estimated structural parameters $\hat{\Theta}$. Let $\hat{\beta}$ be the empirical slope coefficient obtained by running the Campbell-Shiller regressions on our updated dataset. Since we explicitly derive the functional form of the Campbell and Shiller slope coefficients implied by the model as a function of the structural parameters, we can directly test $\mathrm{H}_0 : \beta(\hat{\Theta}) = \hat{\beta}$ using a standard GMM approach, *without resorting to simulation methods.*

Table 5 summarizes the results. We find that both the absolute levels of the slope coefficients and their patterns, as a function of maturity, closely mirror the

Table 5
Campbell and Shiller regressions
This table reports the Campbell and Shiller regressions. The main regression equation is

$$R_{t+m}^{n-m} - R_t^n = \alpha + \beta\left(\frac{m}{n-m}\right)\left(R_t^n - R_t^m\right) + \varepsilon_t,$$

where $R_t^n$ is the yield of a bond with maturity $n$ at time $t$. The expectation hypothesis implies that the coefficient $\beta$ is equal to 1. The value of $m$ is taken to be one month. Panel A presents the results of Campbell and Shiller regressions based on the updated dataset. Panel B presents the values of the same $\beta$ coefficient *implied* by the structural model at the estimated values of the structural parameters. The closed-form solution for the coefficient is given in the appendix. Standard errors are given in parentheses.

| 3 months | 6 months | 1 year | 2 years | 5 years | 10 years |
|---|---|---|---|---|---|
| *Panel A. empirical Campbell and Shiller coefficients,* $\hat{\beta}_{\text{cs}}$ | | | | | |
| 0.216 | −0.33 | −0.93 | −1.22 | −2.21 | −3.67 |
| (0.171) | (0.27) | (0.42) | (0.59) | (0.86) | (1.23) |
| *Panel B. model-implied Campbell and Shiller coefficients,* $\beta(\hat{\Theta})$ | | | | | |
| 0.106 | 0.0183 | −0.1564 | −0.5138 | −1.7677 | −4.9151 |
| *p-values* of $H0 : \beta(\hat{\Theta}) = 1$ | | | | | |
| 0.0000 | 0.0001 | 0.0029 | 0.0051 | 0.0006 | 0.0000 |
| *p-values* of $H0 : \beta(\hat{\Theta}) = \hat{\beta}_{\text{cs}}$ | | | | | |
| 0.2600 | 0.0985 | 0.0327 | 0.1157 | 0.3035 | 0.1557 |

results in Campbell and Shiller. The slope coefficient at a one-year horizon implied by the structural model is −0.15, compared to a value of −0.93 obtained by applying a Campbell and Shiller regression procedure. As the horizon increases, the slope coefficients decrease as in Cambpell and Shiller. At a ten year horizon, the implied slope regression coefficient is −4.91 while the empirical Campbell-Shiller value is −3.67. We run chi-square tests of the null hypothesis that $H0: \beta(\hat{\Theta}) = 1$ and that the two sets of coefficients are equal, i.e., $H0: \beta(\hat{\Theta}) = \hat{\beta}_{cs}$. We find that the implied values of the Campbell-Shiller regression coefficients strongly reject the EH at any confidence level. We also find that, with the exception of the one-year holding period, the implied slope coefficients $\beta(\hat{\Theta})$ are not significantly different from those obtained by Campbell and Shiller. The average $p$-value across the holding period returns is 0.16. We also run a joint chi-square test and do not reject the hypothesis that the coefficients are equal ($p$-value of 0.13).

The model can replicate EH rejection because the risk premium is time varying and state-dependent and because the model-implied equilibrium price of risk is not a constant multiple of the interest rate volatility, as discussed in Section 2.1. But is EH rejection due to time variation of the *nominal* or the *real* risk factors? The question of forward rates as conditionally unbiased predictors of future spot rates has been addressed, among others, by Fama (1976a) and Fama and Bliss (1987). Stambaugh (1988) tests if forward excess returns are unbiased linear predictors of excess holding-period returns. More recently, Bekaert et al. (1997) develop a small-sample test statistic and find strong empirical evidence against a generalized version of the U-EH, allowing for a constant risk premium. They suggest that a reason for rejection of the EH is a peso problem effect due to the inflation process. The U.S. economy has rarely been in a high-inflation regime. A steep term structure, due to expectations of large inflation shocks, might therefore appear excessive from an ex post perspective. They fit and test a regime-switching model of interest rates and find supporting evidence for this explanation.

In our framework, we can directly compare the constant part of the term premium with the part that generates deviations from the EH. Using the overidentifying restrictions given by the structural monetary model, we can extend the previous literature to ask whether EH rejection is due to time variation in the risk premium on nominal (monetary) shocks or real (technological) shocks. We study this issue in three ways: First, we build a formal GMM test of time variation due to the two factors. Second, we decompose the total volatility of the model-implied forward risk premium into two components. Third, we regress the forward premium onto the inflation risk premium to assess its importance in explaining the failure of the EH.

### *7.1. A test of time variation in risk premia*

We develop a formal GMM test to assess whether the two sources of time variation in the forward risk premium are statistically significant. From Eq. (12), a test for the source of uncertainty responsible for violations of the EH can be formalized as follows:

$$\text{H1}: b_v'(\tau) + \Theta_0^v \exp(-k_v\tau) = 0 \text{ and } b_w'(\tau) + \Theta_0^w \exp(-k_w\tau) = 0,$$
$$\text{H2}: b_z'(\tau) + \Theta_0^z \exp(-\xi\tau) = 0.$$

If deviations from the EH are due mainly to nominal shocks, H1 would be rejected, but not H2. Alternatively, if the deviations are due to productivity shocks we would expect the opposite to hold. We construct the following Wald test for the two potential sources of violations:

$$W_i = T[g_i(\theta_T)]' \left\{ \left[ \frac{g_i(\theta)}{\partial\theta}\bigg|_{\theta=\theta_T} \right] \Sigma_i \left[ \frac{g_i(\theta)}{\partial\theta}\bigg|_{\theta=\theta_T} \right]' \right\}^{-1} [g_i(\theta_T)], \quad i = 1, 2.$$

with $g_1(\theta)$ and $g_2(\theta)$ being the two testable restrictions from H1 and H2 and $\Sigma_i$ being a consistent estimator of the covariance matrix of the residuals. Under the null hypothesis, the test statistics $W_i$ are asymptotically chi-squared distributed. Table 6 summarizes the empirical results.

We find that the EH is rejected because of time variation in the risk premium on both nominal (monetary) factors and real (technological) shocks. The $p$-values of the two nominal factors are less than 1% for any maturity except the overnight rate. This result confirms the conjecture that monetary shocks and inflation risk play an important role in EH rejection. However, the evidence also shows that the forward premium is time varying even abstracting from inflation risk. The $p$-values of the real (technological) factor are smaller than 1% for all maturities.

### 7.2. Decomposition of the forward premium

Let $f(t,T) - \mathrm{E}_t[r_T]$ be the forward risk premium. We decompose its volatility into two components: the time variation in $v_t$ and $w_t$ and the time variation in $z_t$. Table 7 illustrates the relative importance of monetary and real factors in driving the time variation of the forward risk premium. Consistent with real business cycle models, we find that real (technological) shocks are the most important factors in driving the forward risk premium. The monetary factor, however, plays an important role. About 43% of the volatility in the one-year forward premium is due to uncertainty about monetary policy. The result is similar when we exclude the 1979–1982 period. We also find that the monetary factor's contribution is not transitory. Even at a ten-year horizon, the percentage of the forward premium volatility due to $v_t$ and $w_t$ is 32%.

### 7.3. The relation between inflation and forward risk premia

We can directly explore the extent to which the inflation risk premium explains deviations from the EH by regressing the forward premium onto the time series of the model-implied one-year inflation risk premium:

$$f(t,T) - \mathrm{E}_t(r_T) = \alpha + \beta \times IRP(t,T) + \varepsilon_t.$$

Table 6
Unbiased expectation hypothesis test
This table presents the results of the unbiased expectations hypothesis (U-EH) tests. U-EH claims that the matching-maturity forward rate is a conditionally unbiased estimator of the expected future spot interest rate. We directly compare the part of the term premium that is constant with the part that generates deviations from the expectations hypothesis. Moreover, using the overidentifying restrictions of the structural monetary model, we can identify the part that is generated purely by nominal shocks. Two null hypothesis are tested: $\mathrm{H1}: b_v'(\tau) + \Theta_0^v \exp(-k_v\tau) = 0$ and $b_w'(\tau) + \Theta_0^w \exp(-k_w\tau) = 0$ and $\mathrm{H2}: b_{z_i}'(\tau) + \Theta_0^{z_i} \exp(-\xi_{z^i}\tau) = 0$. Under H1, the nominal factor does not cause deviations from U-EH. Under H2, the real factor does not cause deviations from the U-EH Hypothesis. The table presents the results for different maturities in the term structure. The table reports the values of the Wald statistics and associated $p$-value.

| | H1<br>Real ($z_t$) | H2<br>Nominal ($v_t$, $w_t$) |
|---|---|---|
| 0 month | 0.00<br>100.00% | 0.00<br>100.00% |
| 1 month | 13.1<br>$<1\%$ | 21<br>$<1\%$ |
| 12 months | 14.6<br>$<1\%$ | 23.2<br>$<1\%$ |
| 3 years | 14.8<br>$<1\%$ | 24.1<br>$<1\%$ |
| 5 years | 15.2<br>$<1\%$ | 25.7<br>$<1\%$ |
| 10 years | 17.1<br>$<1\%$ | 27.4<br>$<1\%$ |

The results of the null hypothesis $\mathrm{H}_0 : \beta = 0$ are summarized in Table 8. At a three-month horizon, we find that the inflation risk premium does not significantly affect the time variation of the forward premium. At horizons of six months and above, however, the contribution of inflation risk is significant and the $p$-values of the test $\mathrm{H}_0 : \beta = 0$ are below 0.01. This result is robust to the exclusion of the 1979–1982 period.

## 8. The term structure of volatility

Completely affine models of the term structure impose severe restrictions on the structure of conditional second moments, as they are perfectly correlated with the price of risk. In our model, a component of monetary policy uncertainty, $w_t$, affects the conditional mean of the stochastic discount factor (thus interest rates) but not its conditional volatility. This factor is not priced: it can affect the term structure of

Table 7
Decomposition of forward premium volatility
This table summarizes the relative percentage contribution of the monetary and real factors on the total volatility of the forward risk premium. The model-implied forward premium is computed in closed form as

$$f(t,T) - \mathrm{E}_t[r_T] = A(\tau) + B_w'(\tau)w_t + B_v'(\tau)v_t + B_{z^i}'(\tau)z_t^i$$

(see Eq. (12) for the exact functional form). We use the delta method to compute the relative volatility contributions.

| Horizon $T-t$ | Contribution to time variation | |
|---|---|---|
| | Monetary (%) | Real (%) |
| 3 months | 28 | 72 |
| 6 months | 43 | 57 |
| 1 year | 43 | 57 |
| 2 years | 42 | 58 |
| 5 years | 41 | 59 |
| 10 years | 32 | 68 |

Table 8
The forward risk premium and the inflation risk premium
The table presents the results of the regression of the forward premium on the inflation risk premium:

$$f(t,T) - \mathrm{E}_t[r_T] = \alpha + \beta\, IRP(t,T) + \varepsilon_t$$

(see Eq. (12) for the exact functional form of forward premium). Column 1 contains the time horizon of the test The remaining columns illustrate the F-test on the significance of the inflation risk premium for the forward premium time variation, the associated $p$-value, and the $R^2$ of the regression.

| $T-t$ | Full sample $H_0: \beta = 0$ | | Excluding 1979–1982 $H_0: \beta = 0$ | |
|---|---|---|---|---|
| | $F$-stat | $P$-val | $F$-stat | $P$-val |
| 3 months | 0.25 | 0.62 | 1.72 | 0.19 |
| 6 months | 4.13 | 0.04 | 6.73 | 0.01 |
| 1 year | 23.10 | 0.00 | 15.91 | 0.00 |
| 2 years | 34.28 | 0.00 | 8.75 | 0.00 |
| 5 years | 128.34 | 0.00 | 62.47 | 0.00 |
| 10 years | 276.99 | 0.00 | 137.19 | 0.00 |

interest rate volatility without affecting expected excess returns. We study the extent to which our model helps explain conditional interest rate volatility. We consider two types of GMM tests. First, we focus on the second moments of the *level* of interest rates. We then study the second moments of yield *changes*.

### 8.1. Second moments of yield curve level

Table 9, Panel A, illustrates the results of a GMM test for the conditional second moment of the yield curve. The empirical three-month yield volatility is 2.68,

Table 9
Volatility of yields (level)

*Panel A. term structure of yield volatility*
This table presents the goodness of fit for the volatility of yields differences. The conditional volatility is given by

$$Var[y(\tau)] = \left[\frac{b_v(w)}{\tau}\right]^2 Var(w) + \left[\frac{b_v(\tau)}{\tau}\right]^2 Var(v) + \left[\frac{b_{z^i}(\tau)}{\tau}\right]^2 Var(z),$$

where $v_t$ and $z_t^i$ are stochastic factors and $b(\tau)$ is a known function of maturity and structural parameters. "Model vol" stands for the unconditional volatility of yields at different maturities. "Data vol" is the sample unconditional volatilities of yields. $\chi^2$-test is a test of the null hypothesis that the volatilities of yields implied by the model are equal to their sample counterpart. The unit of measure is one hundred basis points.

| | 3 months | 6 months | 1 year | 2 years | 5 years | 10 years | Joint test |
|---|---|---|---|---|---|---|---|
| Model vol | 2.32 | 2.30 | 2.27 | 2.24 | 2.19 | 2.15 | |
| *Data vol* | 2.68 | 2.69 | 2.66 | 2.59 | 2.48 | 2.42 | |
| $\chi^2$-test ($p$-value) | 0.31 | 0.26 | 0.23 | 0.24 | 0.43 | 0.08 | 0.31 |

*Panel B. joint test of yield volatility*
This table presents the results of the joint tests of the conditional volatility fitting. We estimate the following restrictions:

$$(y^n_{t+\Delta t} - \mathrm{E}_t[y^n_{t+\Delta t}])^2 = \alpha^n + \beta^n \times \phi + \varepsilon_{t+\Delta t},$$

where $\phi$ is the model-implied conditional second moment. We ask whether the model-implied time-varying conditional second moment is able to track the empirical second moment properties of the data. We test the null hypothesis that $\mathrm{H}_0 : (\alpha^n = 0, \beta^n = 1)$ *for all maturities simultaneously*. The moment conditions are defined as $h_T(x_t, \theta, n) = (y^n_{t+\Delta t} - \mathrm{E}_t[y^n_{t+\Delta t}])^2 - (\alpha^n + \beta^n \times \phi)$ and the test statistics are given by

$$d_T = T[h_T(x_t, \theta)'_{\theta=0} W_T^{-1} h_T(x_t, \theta)_{\theta=0} - h_T(x_t, \hat{\theta})' W_T^{-1} h_T(x_t, \hat{\theta})].$$

| | $\Delta t = 1$ month | $\Delta t = 6$ months | $\Delta t = 1$ year |
|---|---|---|---|
| *Full sample* | | | |
| $d$-stat | 20.78 | 7.54 | 6.23 |
| $p$-value | 0.06 | 0.10 | 0.21 |
| *1982–1998* | | | |
| $d$-stat | 19.44 | 14.13 | 11.96 |
| $p$-value | 0.07 | 0.26 | 0.47 |

compared to a fitted value of 2.32. A GMM test of the null hypothesis that the model-implied term structure of yield volatility is equal to its empirical counterpart is not rejected at any maturity between three months and five years. The $p$-value of a joint test on all maturities is 0.31. The only rejection takes place at a ten-year horizon. At this horizon, the empirical volatility is 2.43 versus a model-implied volatility of 2.15. Although the overall declining pattern of the term structure is captured by the model, it is important to note that the model fails to generate the often observed volatility hump at the six-month horizon.

We also consider the parametric analysis of conditional second moments by Chan et al. (1992) and Duarte (2000). We run the following regression:

$$(y^n_{t+\Delta t} - \mathrm{E}_t[y^n_{t+\Delta t}])^2 = \alpha_n + \beta_n \times \phi + \varepsilon_{t+\Delta t} \tag{13}$$

where $\phi$ is the model-implied variance. We ask whether the model-implied time-varying conditional variance is able to track its empirical counterpart. We test the joint null hypothesis that $\mathrm{H}_0 : \alpha_n = 0$ and $\mathrm{H}_0 : \beta_n = 1$ and we quantify the proportion of the total squared variation in interest rate changes that is explained by the model-implied time series. Let $h_T$ be the cross-sectional vector of average estimation errors:

$$h_T(x_t, \alpha, \beta) = \frac{1}{T}\sum_{t=1}^{T-\Delta t}\left[\left(y^n_{t+\Delta t} - \mathrm{E}_t[y^n_{t+\Delta t}]\right)^2 - (\alpha_n + \beta_n \times \phi)\right]$$

with $t \leqslant T - \Delta t$. We obtain a simultaneous test of the null hypothesis that $\mathrm{H}_0 : (\alpha_n = 0, \beta_n = 1)$ *for all maturities* from

$$d_T = T\left[h_T(x_t, \theta)'_{\theta=0} W_T^{-1} h_T(x_t, \theta)_{\theta=0} - h_T(x_t, \hat{\theta})' W_T^{-1} h_T(x_t, \hat{\theta})\right]$$

which is asymptotically distributed, under the null hypothesis, as a chi-square. The results are summarized in Table 9, Panel B. We consider two sample periods. The first is 1960–2000, corresponding to the sample size used in the estimation of the structural model. The second is 1983–1998, which is used to compare the results with Duarte (2000), who compares the fitting properties of a standard three-factor CIR model with a model in which the price of risk can switch sign. We find that both our structural model and Duarte's reduced-form specification explain conditional second moments better than a standard CIR model. The $p$-value of the joint test $\mathrm{H}_0 : (\alpha_n = 0, \beta_n = 1)$ on all maturities is 0.06 for the full sample and 0.07 for the 1982–1998 (Duarte) subsample. The $p$-value increases as we increase the sampling interval. In the case of the CIR model, the same null hypothesis is easily rejected with $p$-values lower than 0.01. Moreover, for the 1983–1998 period Duarte reports an $R^2$ ranging between 0.07 and 0.15 (this period does not include the years of high interest rate volatility). In the same sample period we obtain an $R^2$ of 0.16.

The key feature of the model that helps explain the conditional second moments is the lack of perfect correlation between the model-implied market price of risk and conditional interest rate volatility. When we reparametrize the model to force the price of risk to be proportional to interest rate volatility, we find similar results to Duffee (2002), Dai and Singleton (2000), and Duarte (2000), who strongly reject the class of completely affine models.

Table 10
Volatility of yields (changes)
This table presents the goodness of fit for the volatility of yields differences. The volatility is given by

$$Var[y_{t+\tau}(m) - y_t(m)] = \left[\frac{b_v(\tau)}{\tau}\right]^2 Var(v_{t+\tau} - v_t) + \sum_{i=1}^{n}\left[\frac{b_{z^i}(\tau)}{\tau}\right]^2 Var(z^i_{t+\tau} - z^i_t)$$

where $v_t$ and $z^i_t$ are stochastic factors and $b(\tau)$ is a known function of maturity and structural parameters. "Model vol" stands for the unconditional volatility of yield differences at different maturities. "Data vol" is the sample unconditional volatilities of yield changes. $\chi^2$-test is a test of the null hypothesis that the volatilities of changes in yields impled by the model are equal to their sample counterparts. The unit of measure is percentage terms.

| | 3 months | 6 months | 1 year | 2 years | 5 years | 10 years | Joint test |
|---|---|---|---|---|---|---|---|
| Model vol | 0.367 | 0.334 | 0.303 | 0.275 | 0.255 | 0.322 | |
| Data vol | 0.575 | 0.559 | 0.541 | 0.486 | 0.389 | 0.328 | |
| $\chi^2$-test ($p$-value) | 0.043 | 0.044 | 0.022 | 0.012 | 0.005 | 0.809 | 0.052 |

### 8.2. Second moments of yield curve changes

Tests of the model's ability to reproduce the volatility of yield *changes* are summarized in Table 10, Panel B. In five of nine cases, we reject the null hypothesis. In absolute terms, at a five-year horizon, the model-implied volatility of yield changes is 0.255 versus an empirical volatility of 0.389. The performance of the model is significantly better for the ten-year yield to maturity. These results show the boundaries of the structural affine model's ability to explain the empirical properties of interest rates. The linear specification of the local volatility in the factors, even when coupled with a relatively flexible specification of the price of risk, cannot fully describe the second moments of yield changes. Unfortunately, nonlinear specifications of the volatility are not analytically tractable.

## 9. Other macroeconomic variables

### 9.1. Real interest rates

Fig. 2, Panel A, illustrates the dynamics of the estimated short-term real interest rate. During the sample period 1960–2000, it ranges between −2% and 5%. The average short-term real rate is 2%, while the average long-term real rate is 2.5%. The correlation between short- and long-maturity real rates is small. The volatility of short-term real rates is substantially higher than the volatility of long-term real rates, which fluctuates around 2.5%. Table 3 shows the correlation matrix for the estimated values of the nominal interest rate, the real interest rate, the risk premium, and the expected inflation rate.

One of the most significant results is the negative correlation between the real interest rate and expected inflation, which is $-49\%$ at a ten year horizon. This compares to a 61% correlation coefficient between the nominal interest rate and inflation. We also find that the inflation risk premium is positively correlated with the level of inflation, with a correlation coefficient of 44% at a ten-year horizon, and negatively correlated with the real interest rate.

## 9.2. The expected inflation rate

Let $\mathrm{E}_t(\pi_{t+1}|I_t)$ be model-implied inflation. If the model is correctly specified, then the prediction errors are orthogonal to any function of $x_t$, measurable with respect to $I_t$. If the model is not correctly specified, one could improve the model forecast errors using some function of the explanatory variable $\phi(x_t)$, i.e., $\mathrm{E}_t(\pi_{t+1}|I_t) + \theta'\phi(x_t)$. Thus, let the inflation forecast error be $u_{t+1} = \pi_{t+1} - \mathrm{E}_t(\pi_{t+1}|I_t) - \theta'\phi(x_t)$ and let us study the null hypothesis $\mathrm{H}_0 : \theta = 0$ using a conditional GMM test.

We compare the model-implied expected inflation with publicly available inflation forecasts provided by the Federal Reserve Bank of Philadelphia and the University of Michigan. The data from the Fed of Philadelphia consists of the expected price change for the following four quarters and are available at a quarterly frequency. The value is calculated as the median value from the Survey of Professional Forecasters compiled by the Fed of Philadelphia. Since several popular macro models assume that the inflation rate follows a random walk, we also compare the performance of the structural model with respect to the random walk hypothesis.

Table 11 summarizes the results. At a 12-month horizon, the null hypothesis that the survey-based prediction errors are orthogonal to lagged inflation is strongly rejected in the case of both the Philadelphia Fed and the University of Michigan, with $p$-values less than 1%. At a 12-month horizon, the structural model outperforms all other inflation forecasts in terms of $p$-values. The structural model is the only one to survive the orthogonality test. We strongly reject the null hypothesis of orthogonality in the case of the forecasts provided by both the University of Michigan and the Fed of Philadelphia. At a 12-month horizon, the structural model is not rejected based on the same null hypothesis. At higher frequencies, such as a one-month horizon, the model finds it more difficult to fit inflation. However, the null hypothesis that lagged values of inflation are orthogonal to prediction errors is rejected much less strongly than in the case of the random walk specification; the $d_T$ statistic is equal to 18.33 versus 75.37. The difficulty of predicting inflation at a one-month frequency is well known and supported by the fact that neither the University of Michigan nor the Federal Reserve Bank of Philadelphia offers inflation forecasts at a one-month frequency.

The unconditional mean of the term structure of expected inflation is downward sloping. This suggests mean-reverting behavior of the inflation rate, consistent with a monetary supply process targeting deviations from a long-term inflation objective.

Table 11
Orthogonality tests of expected inflation
The table presents the results of the orthogonality test for the inflation forecast. Let $u_{t+1}$ be the prediction error for the inflation rate from the structural model and from (a) the Federal Reserve Bank of Philadelphia, (b) the survey data from The University of Michigan, and (c) the random walk model. We study the extent to which the inflation forecasts are orthogonal to lagged explanatory variables by testing the null hypothesis $H_0 : \theta = 0$ in a GMM framework with the following moment restrictions:

$$\begin{bmatrix} u_{t+1}(\theta) \\ u_{t+1}(\theta) \otimes [\phi(x_t)] \end{bmatrix}$$

with the unrestricted prediction errors defined as

$$u_{t+1} = \pi_{t+1} - \mathrm{E}_t(\pi_{t+1}|I_t) - \theta' \phi(x_t).$$

We test the null hypothesis using the following statistics $d_T$

$$d_T = T[h_T(x_t, \theta(\mathrm{H}_0))' W_T^{-1} h_T(x_t, \theta(\mathrm{H}_0)) - h_T(x_t, \theta^*)' W_T^{-1} h_T(x_t, \theta^*)]$$

which is $\chi^2$ distributed under the null hypothesis. The $p$-value associated to the $d_T$ statistics and of the restricted parameters are in parentheses. We consider the following lagged explanatory variables:

$$\phi(x_t) = \begin{bmatrix} \text{const} \\ \pi_t \\ \pi_t^2 \end{bmatrix}.$$

| | Source of forecast | | | | | |
|---|---|---|---|---|---|---|
| | Model | Random walk | Model | Random walk | Philadelphia | Michigan |
| | Forecast horizon | | | | | |
| | 1 month | | 1 year | | | |
| | $t, t+1$ | $t, t+1$ | $t, t+12$ | $t, t+12$ | $t, t+12$ | $t, t+12$ |
| Constant | 0.350 | 0.170 | 0.240 | 0.424 | −1.178 | −2.165 |
| | (0.345) | (0.000) | (0.485) | (0.280) | (0.006) | (0.000) |
| $\pi$ | −0.541 | −0.539 | −0.316 | 0.062 | 0.225 | 0.826 |
| | (0.119) | (0.000) | (0.302) | (0.459) | (0.170) | (0.000) |
| $\pi^2$ | 0.074 | 0.117 | 0.038 | −0.022 | −0.010 | −0.057 |
| | (0.011) | (0.266) | (0.171) | (0.156) | (0.396) | (0.026) |
| $d_T$ | 18.328 | 75.369 | 3.024 | 11.905 | 20.858 | 44.330 |
| | (0.001) | (0.000) | (0.434) | (0.010) | (0.000) | (0.000) |

### 9.3. Monetary holdings

The model-implied unconditional expected value of the money growth rate is 6.81% with respect to an empirical value of 6.26%.[16] We run a GMM test and do not reject the null hypothesis that the difference is zero. We repeat the previous

[16]A table reporting detailed results of asymptotic test statistics is available from the authors upon request.

conditional analysis for the monetary process. Letting the prediction error be $u_{t+12} = \ln\left(\frac{M_{t+12}}{M_t}\right) - \mathrm{E}_t\left[\ln(\frac{M_{t+12}}{M_t})\right]$, we estimate and test whether $u_{t+12}$ is orthogonal to lagged values of the explanatory variables, i.e., $\mathrm{E}\left[u_{t+12} \otimes \phi(\frac{M_t}{M_{t-12}})\right] = 0$. At a one-month frequency, we reject the orthogonality hypothesis with a $p$-value of 4%. However, at a one-year frequency, the coefficients on lagged values of money are not, at the individual level, significantly different from zero, with a $p$-value above 25%. This suggests that the model can capture the dynamics of the money supply at a one-year frequency at statistically acceptable levels.[17]

## 10. Conclusions

In the theoretical part of the paper, we first show the link between a monetary version of a real business cycle model with taxes on nominal profits and the latest generation of (essentially) affine term structure models (i.e., Duffee, 2002; Dai and Singleton, 2000) in which the price of risk is not a constant multiple of interest rate volatility. Second, we characterize a structural model in which the inflation risk premium is positive and time varying. The source of this premium is the fiscal system. When taxes are computed on nominal income, the (after-tax) real return on capital is affected by inflation shocks. Thus, inflation can generate distortions in real capital accumulation and in bond yields carry a time-varying inflation risk premium. The monetary policy rule is endogenous and similar in spirit, but not isomorphic, to a Taylor (1993) rule. The monetary authority adjusts the money supply based on deviations of inflation and output growth from their long-term objectives. We characterize the equilibrium and obtain closed-form solutions for the real and nominal term structures of interest rates and for the inflation risk premium.

In the empirical section of the paper, we estimate the structural parameters of the economy using panel data on U.S. Treasury bonds ranging from 1960 to 2000 with maturities from one month to ten years. From the empirical analysis we learn the following:

(a) The model generates both cross-sectional and forecasting errors that are smaller than completely affine specifications. These results are consistent with the findings in Duffee (2002) with regard to more flexible specifications of the market price of risk.
(b) We do not reject the null hypothesis that the model-implied term structure of volatility is correctly specified. However, the model fails to reproduce the volatility curve hump at the six month maturity. We also find that the model gives mixed results in fitting the conditional volatility of yield *changes*.
(c) We reject the null hypothesis that the model-implied forward risk premium is constant, consistent with empirical rejections of the expectation hypothesis. We find that 43% of the volatility of the forward risk premium is due to monetary shocks.

[17]The table with the detailed results are available upon request.

(d) The sign and size of the model-implied linear projection coefficients are consistent with those found by Campbell and Shiller (1991). They are negative and downward sloping. Moreover, we do not reject the null hypothesis that the model-implied and empirical Campbell-Shiller regression coefficients are equal.
(e) The slope of the inflation risk premium term structure is sharply upward increasing. The one-month interest rate carries a small inflation risk premium while the average ten-year inflation risk premium is 70 basis points. At medium- and long-term horizons, the Fisher hypothesis is strongly rejected by the data.
(f) The inflation risk premium is time varying. The ten-year premium has fluctuated between 20 and 140 basis points and is positively correlated with the actual level and volatility of inflation.
(g) We use the model to separate the relative contribution of nominal and real factors in the time variation of the inflation risk premium and in deviations from the expectation hypothesis. We find that the inflation risk premium can explain 23% (42%) of the time variation of the five(ten)-year forward risk premium.

## Appendix A

### A.1. Dynamic capital accumulation equation

The representative consumer owns the company and decides how much to invest or consume. The explicit cost of capital is zero in the sense that holding the capital does not imply any cash outflows as in the case of borrowed capital. However, there are two types of costs associated with the production technology: depreciation and variable costs. Both of them are deductible for tax purposes. The capital gains tax is levied on the increase in the total nominal value of capital. If inflation does not change, then the capital gains tax is zero.

$$\frac{\mathrm{d}K_t}{K_t} = \left[ -\frac{C_t}{K_t}\,\mathrm{d}t + \mathrm{d}Y_t(1-\tau_{\mathrm{pr}})(1-\lambda_s) - [\lambda_m(1-\tau_{\mathrm{pr}}) - \ell]\,\mathrm{d}t \right.$$
$$\left. -\tau_{\mathrm{pr}}\lambda_s\, COV_t\left(\frac{\mathrm{d}p_t}{p_t}, \mathrm{d}Y_t\right)\mathrm{d}t - \tau_{\mathrm{cg}}\frac{\mathrm{d}p_t}{p_t} + \tau_{\mathrm{cg}}\sigma_P^2(\cdot)\,\mathrm{d}t - \frac{M_t^d\,\mathrm{d}t}{K_t}\right]. \tag{A.1}$$

The optimal investment–consumption plan is given by

$$\max_{\{C_t, M_t^d\}} \mathrm{E}_0\left[\int_0^\infty \mathrm{e}^{-\rho t}\left[\ln(C_t) + \gamma\,\ln(M_t^d)\right]\mathrm{d}t\right]$$

subject to the budget constraint (A.1).

In order to simplify the notation, let us define the following parameters:

$$A = -[\lambda_m(1-\tau_{\mathrm{pr}}) - \ell], \quad A_{py} = -\tau_{\mathrm{pr}}\lambda_s,$$

$$A_z = (1-\lambda_s)(1-\tau_{\mathrm{pr}})\mu_{y^i}, \quad A_c = A_m = 1,$$

$$A_p^{\mu} = -\tau_{\rm cg}, \quad B_z = \sigma_{y^i},$$

$$A_p^{\sigma} = \tau_{\rm cg}, \quad B_p = -\tau_{\rm cg}. \tag{A.2}$$

Then the capital accumulation equation can be written in the following form:

$$\frac{{\rm d}K_t}{K_t} = \left[A + A_z z_t + A_p^{\mu}\mu_p(\cdot) + A_p^{\sigma}\sigma_p^2(\cdot) + A_{py}\, cov\left(\frac{{\rm d}p_t}{p_t}, {\rm d}Y_t^i\right)\right.$$

$$\left. -A_c\,\frac{C_t}{K_t} - A_m\,\frac{M_t^d}{K_t}\right]{\rm d}t \tag{A.3}$$

$$+\left[B_z\sqrt{z_t^i}\,{\rm d}W_t^{y^i} + B_p\sigma_p(\cdot)\,{\rm d}W_t^p\right], \tag{A.4}$$

where $z_t^i$ are underlying real factors driving the productivity of capital. Let us assume that there exists an equilibrium price process that takes the form $\frac{{\rm d}p_t^*}{p_t^*} = \mu_{p*}(\cdot)\,{\rm d}t + \sigma_{p*}(\cdot)\,{\rm d}W_t^{p^*}$. We will later verify that this is indeed the case and solve for the market clearing functional values of $\mu_{p*}$ and $\sigma_{p*}$. In equilibrium, there must exist a value function $J(t, K_t, z_t, v_t, w_t)$ and control variables $\{C_t, M_t^d\}$ such that the following Benvenieste-Scheinkman condition is satisfied

$$-\frac{\partial}{\partial t}\,J(\cdot) = \max_{\{C_t, M_t^d\}}\left[{\rm e}^{-\rho t}\ln(C_t) + {\rm e}^{-\rho t}\gamma\,\ln(M_t^d) + \mathscr{A}J(\cdot)\right]$$

where $\mathscr{A}J$ is the differential operator applied to the function $J(\cdot)$. Let us consider the following guess for $J(\cdot)$

$$\left[\frac{1}{\rho}\,{\rm e}^{-\rho t}\right]^{-1} J(t, K_t, z_t^i, v_t, w_t) = P + [Q\,\ln(\rho K_t) + R_{z^i}z_t^i + R_v v_t + R_w w_t] - \frac{\partial}{\partial t}\,J(\cdot)$$

$$= {\rm e}^{-\rho t}(P + [Q\,\ln(\rho K_t) + R_{z^i}z_t^i + R_v v_t + R_w w_t]).$$

Let us compute the differential $\mathscr{A}J(\cdot)$. Since the function $J(\cdot)$ is linear with respect to the state variables $[z_t, v_t, w_t]$, all the second derivatives are zero. Thus

$$\begin{aligned}
\mathscr{A}J(t, K_t, z_t, v_t, w_t) &= \frac{\partial J}{\partial K_t}\,\mu_{K_t} + \frac{\partial J}{\partial z_t^i}\,\mu_{z_t^i} + \frac{\partial J}{\partial v_t}\,\mu_{v_t} + \frac{\partial J}{\partial w_t}\,\mu_{w_t} + \frac{1}{2}\,\frac{\partial^2 J}{\partial K_t}\,\sigma_K^2 \\
&= Q\,\frac{1}{K_t}\,\mu_K + R_{z^i}\mu_{z^i} + R_v\mu_v + R_{\omega}\mu_{\omega} + \frac{1}{2}\left(-Q\,\frac{1}{K^2}\right)\sigma_K^2 \\
&= Q\left[A + A_z z_t + A_p^{\mu}\mu_p(\cdot) + A_p^{\sigma}\sigma_p^2(\cdot) + A_{pk}\,cov\left(\frac{{\rm d}p_t}{p_t}, \frac{{\rm d}K_t}{K_t}\right)\right. \\
&\quad \left. -A_c\,\frac{C_t}{K_t} - A_m\,\frac{m_t}{K_t}\right] + R_{z^i}(\xi^i z_t^i + \zeta^i) \\
&\quad + R_v(k_v v_t + \theta_v) + R_{\omega}(k_{\omega}\omega_t + \theta_{\omega}) - \frac{Q}{2}\left(B_z^2 z_t^i\,{\rm d}t\right. \\
&\quad \left. +B_p^2\sigma_p^2(\cdot)\,{\rm d}t + COV_t\left[B_z\sqrt{z_t^i}\,{\rm d}W_t^{y^i}, B_p\sigma_p(\cdot)\,{\rm d}W_t^p\right]\right)
\end{aligned}$$

where the functional forms $A, A_z, A_p^\mu, A_p^\sigma, B_z$, and $B_p$ are defined in (A.2). The first-order conditions are

$$[C_t] : \mathrm{e}^{-\rho t}\,\frac{1}{C_t} - A_c\left[\frac{1}{\rho}\,\mathrm{e}^{-\rho t}\right] Q\,\frac{1}{K_t} = 0,$$

$$[M_t^d] : \mathrm{e}^{-\rho t}\,\frac{\gamma}{M_t^d} - A_m\left[\frac{1}{\rho}\,\mathrm{e}^{-\rho t}\right] Q\,\frac{1}{K_t} = 0.$$

From these conditions we obtain that the consumption and real money holdings are linear functions of total capital $K_t$.

$$C_t = A_c\,\frac{\rho}{Q}\,K_t, \quad M_t^d = A_m\,\frac{\gamma\rho}{Q} K_t.$$

Let us solve for $Q$. Substitute the optimal policy functions in the Benvenieste-Scheinkman conditions:

$$-\frac{\partial}{\partial t}\,J(\cdot) = \max_{(C_t, M_t^d)}\left[\mathrm{e}^{-\rho t}\,\ln(C_t) + \mathrm{e}^{-\rho t}\,\gamma\ln(M_t^d) + \mathscr{A}J(\cdot)\right],$$

$$\begin{aligned}
&\mathrm{e}^{-\rho t}[P + Q\,\ln(\rho K_t) + R_{z^i} z_t^i + R_v v_t + R_w w_t] \\
&\quad = \mathrm{e}^{-\rho t}\,\ln\left[\frac{\rho}{Q}\,K_t\right] + \gamma\mathrm{e}^{-\rho t}\,\ln\left[\frac{\gamma\rho}{Q}\,K_t\right] + \left[\frac{1}{\rho}\,\mathrm{e}^{-\rho t}\right] \\
&\left(\begin{array}{c} Q\left[A + A_z z_t + A_p^\mu \mu_p(\cdot) + A_p^\sigma \sigma_p^2(\cdot) + A_{py}\,cov\left(\frac{\mathrm{d}p_t}{p_t}, \mathrm{d}y_t^i\right) - A_c\,\frac{\rho}{Q} - A_m\,\frac{\gamma\rho}{Q}\right] + R_{z^i}(\xi^i z_t^i + \zeta^i) + R_v(k_v v_t + \theta_v) \\ +R_w(k_w w_t + \theta_w) - \frac{Q}{2}\left(B_z^2 z_t^i + B_p^2 \sigma_p^2(\cdot) + COV_t\left[B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i}, B_p\sigma_p(\cdot)\,\mathrm{d}W_t^p\right]\right) \end{array}\right)
\end{aligned}$$

The parameter values for $[P, Q, R_{z^i}, R_v, R_w]$ can be solved by matching the coefficient of the state variables [constant, $\ln(K), z_t^i, v_t, w_t$] of the Benvenieste-Scheinkman optimality condition. One can notice that a solution exists since, as proved later, in equilibrium $\mu_p(\cdot)$, $\sigma_p^2(\cdot)$, and $COV_t[B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i}, B_p\sigma_p(\cdot)\,\mathrm{d}W_t^p]$ are affine functions of the underlying factors. Matching the coefficients of $\ln(K)$ we can solve for $Q$:

$$\mathrm{e}^{-\rho t} Q = \mathrm{e}^{-\rho t} + \gamma\mathrm{e}^{-\rho t}.$$

We obtain $Q = (1+\gamma)$. It follows that the optimal policy functions are

$$C_t = A_c\,\frac{\rho}{(1+\gamma)}\,K_t, \quad M_t^d = A_m\,\frac{\gamma\rho}{(1+\gamma)}\,K_t. \tag{A.5}$$

In order to solve for the other parameters and verify the guess for the indirect utility function, we need to solve for the equilibrium price process $\frac{\mathrm{d}p^*}{p^*}$. This can be obtained from the market clearing condition for monetary holdings:

$$p_t^* M_t^{*d} = M_t^s. \tag{MCC}$$

Assuming that markets cleared at time $t=0$, i.e., $p_0^* M_0^d = M_0^s$, by Ito's Lemma, the previous market clearing condition is equivalent to

$$M_t^d\,\mathrm{d}p_t^* + p_t\,\mathrm{d}M_t^d + COV_t(\mathrm{d}p_t^*, \mathrm{d}M_t^d) = \mathrm{d}M_t^s.$$

Equivalently, dividing by $M_t^d$, and substituting the market clearing condition (MCC),

$$\frac{\mathrm{d}p_t^*}{p_t^*} = \frac{\mathrm{d}M_t^s}{M_t^s} - \frac{\mathrm{d}K_t^*}{K_t^*} - COV_t\left(\frac{\mathrm{d}p_t^*}{p_t}, \frac{\mathrm{d}K_t^*}{K_t^*}\right).$$

Substituting $\frac{\mathrm{d}M_t^s}{M_t^s}$ in the previous equation, $[\frac{\mathrm{d}p_t^*}{p_t^*}; \frac{\mathrm{d}K_t^*}{K_t^*}]'$ form a system of two stochastic differential equations driven by the same basis of Brownian motions $[W_t^{y^i}, W_t^M]'$

$$\frac{\mathrm{d}p_t^*}{p_t^*} = \psi_p^*(\cdot)\,\mathrm{d}t + (q_1 - 1)\frac{\mathrm{d}K_t^*}{K_t^*} + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,\mathrm{d}W_t^M,$$

$$\frac{\mathrm{d}K_t^*}{K_t^*} = \psi_K^*(\cdot)\,\mathrm{d}t + B_z\sqrt{z_t^i}\,\mathrm{d}W_t^{y^i} + B_p\sigma_p^*(\cdot)\,\mathrm{d}W_t^{p^*}.$$

The system in not in reduced form. We can express $\left[\frac{\mathrm{d}p_t^*}{p_t^*}; \frac{\mathrm{d}K_t^*}{K_t^*}\right]'$ in reduced form, in terms of the basis of Brownian motions $\left[W_t^{y^i}, W_t^M\right]'$, and obtain the following representation:

$$\frac{\mathrm{d}p_t^*}{p_t^*} = \mu_{p^*}(\cdot)\,\mathrm{d}t + \sigma_{p^*}(\cdot)[W_t^{y^i}, W_t^M]',$$

$$\frac{\mathrm{d}K_t^*}{K_t^*} = \mu_{K^*}(\cdot)\,\mathrm{d}t + \sigma_{K^*}(\cdot)[W_t^{y^i}, W_t^M]'.$$

First solving for $\sigma_K^*$ and $\sigma_p^*$ by substitution, then computing $COV_t\left(\frac{\mathrm{d}p_t^*}{p_t}, \frac{\mathrm{d}K_t^*}{K_t^*}\right)$, and finally solving for the drifts $\mu_K^*\,\mathrm{d}t$ and $\mu_p^*\,\mathrm{d}t$ we have

$$COV_t\left(\frac{\mathrm{d}p_t^*}{p_t^*}, \frac{\mathrm{d}K_t^*}{K_t^*}\right) = \left[\frac{1}{(1-q_2) + B_p(1-q_1)}\right]^2 \times[(q_1-1)(1-q_2)B_z^2 z_t^i + B_p(\sigma_{0M}^2 + \sigma_{1M}^2 v_t)],$$

$$A_{py}\,cov\left(\frac{\mathrm{d}p_t}{p_t}, \mathrm{d}y_t^i\right) = \frac{(q_1-1)(1-q_2)B_z}{(1-q_2) + B_p(1-q_1)}\sigma_{y^i} z_t^i\,\mathrm{d}t,$$

$$\mu_K^* = \frac{A - \frac{A_p^\mu(q_1\bar{\kappa} + q_2\bar{\pi})}{(1-q_2)} - \rho\frac{A_c + A_m\gamma}{(1+\gamma)} + A_z z_t + \frac{A_p^\mu}{(1-q_2)}w_t + COV_t\left(\frac{\mathrm{d}p_t^*}{p_t}, \frac{\mathrm{d}K_t^*}{K_t^*}\right)\left(A_{pk} - \frac{A_p^\mu}{(1-q_2)}\right) + A_p^\sigma\sigma_p^2(\cdot)}{1 + \frac{A_p^\mu(1-q_1)}{1-q_2}},$$

$$\mu_p^* = \frac{1}{(1-q_2)} w_t + \frac{(q_1-1)}{(1-q_2)} \mu_K^* - \frac{(q_1\bar{\kappa} + q_2\bar{\pi})}{(1-q_2)} - \frac{COV_t\left(\frac{dp_t^*}{p_t}, \frac{dK_t^*}{K_t^*}\right)}{(1-q_2)},$$

$$\sigma_K^*[W_t^{y^i}, W_t^M]' = \frac{1}{(1-q_2)+B_p(1-q_1)}\left[(1-q_2)B_z\sqrt{z_t^i}\,dW_t^{y^i} + B_p\sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,dW_t^M\right],$$

$$\sigma_p^*(\cdot)[W_t^{y^i}, W_t^M]' = \frac{1}{(1-q_2)+B_p(1-q_1)}\left[(q_1-1)B_z\sqrt{z_t^i}\,dW_t^{y^i} + \sqrt{\sigma_{0M}^2 + \sigma_{1M}^2 v_t}\,dW_t^M\right].$$

After we have expressions for equilibrium price processes we can verify the original guess for the value function. Substitute the optimal demand functions into the Bellman-Hamilton-Jacobi equation:

$$-\frac{\partial}{\partial t} J(K_t, z_t, v_t, w_t) = \max_{\{C_t, M_t^d\}} \left[e^{-\rho t}\ln(C_t) + e^{-\rho t}\gamma \ln(M_t^d) + \mathscr{A} J(K_t, z_t, v_t, w_t)\right].$$

Substituting the equilibrium value of the drift and volatility of the priceprocess and matching the coefficients of $\{const., \ln K_t, z_t^i, v_t, w_t\}$, it is possible to obtain a linear system of four equations in four unknown $\{P, Q, R_i, R_v\}$. These solutions are independent $K_t$ and the other state variables $\{\mathbf{z}_t, v_t, w_t\}$. Thus, the guess is verified.

**Lemma** (*Feynman-Kac Theorem for unbounded functions, Karatsas-Shreve*). *Consider the diffusion process*

$$dx_t = \mu(t, x_t)\,dt + \sigma(t, x_t)\,dW_t \tag{A.6}$$

*and the function $f(t, x_t)$.The following lemma, by Karatsas and Shreve, generalizes the standard Feynman-Kac theorem to unbounded functions $f(t, x_t)$and nonstationary processes $x_t$:*

(A1) *Let the parameters governing the vector diffusion process $x_t$ be continuous and twice continuously differentiable functions in $x_t$ and let $dx_t$ ave a weak solution $x_t \in R_{++}^k$ unique in probability law.*

*Let $f(t, x_t)$ be bounded below: $f(x_t) \geqslant 0,\ \forall x_t \in R_{++}^k$*

(A2) *Let $\phi(t, x_t) : [0, T] \times R_{++}^k \to R^1$ be continuous and of class $C^{1,2}$ on $[0, T] \times R_{++}^k$.*

(A3) *Let the function $\phi(t, x_t)$ be such that*

$$\max_{0 \leqslant t \leqslant t+s} |\phi(t, x)| \leqslant M(1 + \|x\|^{2\mu}) \tag{A.7}$$

*for some constant $M > 0$ and $\mu \geqslant 1$.*

*If $\phi(t, x_t)$ satisfies the problem*

$$-\frac{\partial \phi(t, x_t)}{\partial t} = \mathscr{A}\phi(t, x_t), \tag{A.8}$$

$$\text{s.t.} \quad \phi(T, x_T) = f(T, x_T) \tag{A.9}$$

*where $A\phi$ is the second-order differential operator applied to the function $\phi$, then $\phi(t, x_t)$ admits the stochastic representation*:

$$\phi(t, x_t) = \mathrm{E}_{t, x_t} f(T, x_T). \tag{A.10}$$

## *A.2. The term structure of nominal interest rates*

Let $\kappa_t^* = (1 + \gamma) \ln K_t^* + \rho t$. Recalling that the equilibrium diffusion process for $\mathrm{d}K_t^*/K_t^*$ is

$$\frac{\mathrm{d}K_t^*}{K_t^*} = \mu_K^* \, \mathrm{d}t + \sigma_K^* \, \mathrm{d}W_t^K$$

we can use Ito's Lemma to compute $\mathrm{d}\kappa_t^* \equiv \mathrm{d} \ln K_t^* = \mu_{\kappa^*} \, \mathrm{d}t + \sigma_{\kappa^*} \, \mathrm{d}W_t^{\kappa^*}$

$$\mathrm{d}\kappa_t^* = \left[(1 + \gamma)\left[\mu_K^* - \frac{1}{2} \, Trace(\boldsymbol{\sigma}_K^* \boldsymbol{\sigma}_K^{*\prime})\right] + \rho\right] \mathrm{d}t + (1 + \gamma)\boldsymbol{\sigma}_K^{*\prime} \, \mathrm{d}\mathbf{W}_t^{K^*}.$$

From the standard first-order conditions of the representative agent, we know that the real price of a zero-coupon bond with a nominal payoff equal to a unit of the numeraire is equal, in equilibrium, to the conditional expected value of the product of the intertemporal marginal rate of substitution times the real payoff of the financial asset:

$$\begin{aligned}
\frac{1}{p_t^*} B_t^\tau &= \mathrm{E}_t\left[\mathrm{e}^{-\rho\tau} \, \frac{\exp(-\ln X_{t+\tau}^*)}{\exp(-\ln X_t^*)} \, \frac{1}{p_{t+\tau}^*}\right] \\
&= \mathrm{E}_t\left[\mathrm{e}^{-\rho\tau} \, \frac{\exp(-(1+\gamma)\ln K_{t+\tau}^*)\frac{1}{p_{t+\tau}}}{\exp(-(1+\gamma)\ln K_t^*) \, \frac{1}{p_t}}\right] \\
&= \mathrm{E}_t\left[\frac{\exp(-((1+\gamma)\ln K_{t+\tau}^* + \rho(t+\tau)))\frac{1}{p_{t+\tau}}}{\exp(-((1+\gamma)\ln K_t^* + \rho t))\frac{1}{p_t}}\right] \\
&= \frac{1}{\exp(-\kappa_t^*)\frac{1}{p_t}} \mathrm{E}_t\left[\exp(-\kappa_{t+\tau}^*)\frac{1}{p_{t+\tau}}\right].
\end{aligned} \tag{A.11}$$

Let $\phi(\kappa_t^*, p_t^*, v_t, z_t; \tau)$ be the solution of the stochastic problem:

$$\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \mathrm{E}_t\left[\exp(-\kappa_{t+\tau}^*)\frac{1}{p_{t+\tau}^*}\right]. \tag{A.12}$$

Since the economy has a constant return to scale production process and logarithmic preferences, let us consider the following log-linear guess:

$$\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \left[\exp(-\kappa_t^*)\frac{1}{p_t^*}\right] A(\tau) \exp\left[-b_v(\tau)v_t - b_w(\tau)w_t - \sum_{i=1}^{n} b_{z^i}(\tau)z_t^i\right]. \tag{A.13}$$

If $2\theta \geqslant \sigma_w^2$, then zero is an unattainable boundary for $w_t$ and $p_t$ and $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t; \tau)$ satisfies assumption (A2) and the polynomial growth condition (A3) of Lemma 1. By the Yamada-Watanabe theorem, it is possible to show that on $R_+^2$ the vector diffusion process $[p_s, v_s]$ has a weak solution for any initial condition $[t; p_t, v_t] \in [0, T] \times R_{++}^2$ so that it satisfies condition (A1) of Lemma 1. $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t; \tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, from which we know that if $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t; \tau)$ is a solution to the stochastic problem (A.11), then it must also be a solution of the following differential problem:

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \mathscr{A}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau), \tag{A.14}$$

$$\text{s.t.} \quad \lim_{\tau \to 0} \phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \left[\exp(-\kappa_t^*)\frac{1}{p_t^*}\right].$$

The left-hand side of (A.14) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi^* = \left[\frac{A'(\tau)}{A(\tau)} - b_v'(\tau)v_t - b_w'(\tau)w_t - \sum_{i=1}^{n} b_{z^i}'(\tau)z_t^i\right]\phi^*. \tag{A.15}$$

Applying Ito's Lemma, the right-hand side of (A.14) is

$$\begin{aligned}
\mathscr{A}\phi(\kappa_t, p_t, v_t, w_t, \mathbf{z}_t; \tau) = {} & \frac{\partial\phi}{\partial\kappa}\mu_{\kappa*} + \frac{\partial\phi}{\partial p}\mu_{p*} + \frac{\partial\phi}{\partial v}\mu_v + \frac{\partial\phi}{\partial w}\mu_w + \sum_{i=1}^{n}\frac{\partial\phi}{\partial z^i}\mu_{z^i} \\
& + \frac{1}{2}\left[\frac{\partial^2\phi}{\partial\kappa^2}\sigma_{\kappa*}^2 + \frac{\partial^2\phi}{\partial p^2}\sigma_{p*}^2 + \frac{\partial^2\phi}{\partial w^2}(\sigma_{0w}^2 + \sigma_{1w}^2 w_t)\right. \\
& \left. + \frac{\partial^2\phi}{\partial v^2}(\sigma_{0v}^2 + \sigma_{1v}^2 v_t) + \sum_{i=1}^{n}\frac{\partial^2\phi}{\partial z^{i2}}\sigma_{z^i}^2 z_t\right] \\
& + \left[\frac{\partial^2\phi}{\partial\kappa\partial p} Cov(\mathrm{d}\kappa^*, \mathrm{d}p^*) + \frac{\partial^2\phi}{\partial\kappa\partial v} Cov(\mathrm{d}\kappa^*, \mathrm{d}v)\right. \\
& \left. + \frac{\partial^2\phi}{\partial\kappa\partial w} Cov(\mathrm{d}\kappa^*, \mathrm{d}w) + \sum_{i=1}^{n}\frac{\partial^2\phi}{\partial\kappa\partial z^i} Cov(\mathrm{d}\kappa^*, \mathrm{d}z^i)\right] \\
& + \left[\frac{\partial^2\phi}{\partial p\partial v} Cov(\mathrm{d}p^*, \mathrm{d}v) + \frac{\partial^2\phi}{\partial p\partial w} Cov(\mathrm{d}p^*, \mathrm{d}w)\right. \\
& + \frac{\partial^2\phi}{\partial v\partial w} Cov(\mathrm{d}v, \mathrm{d}w)
\end{aligned}$$

$$+\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial p\partial z^i}\,Cov(\mathrm{d}p^*,\mathrm{d}z^i)$$

$$\left.+\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial z^i\partial v}\,Cov(\mathrm{d}z^i,\mathrm{d}w)\right]. \tag{A.16}$$

Observe that

$$\frac{\partial\phi}{\partial\kappa^*}=-\phi,\quad \frac{\partial\phi}{\partial p^*}=-\frac{1}{p^*}\phi,\quad \frac{\partial\phi}{\partial v_t}=-b_v(\tau)\phi,\quad \frac{\partial\phi}{\partial w_t}=-b_w(\tau)\phi,\quad \frac{\partial\phi}{\partial z_t^i}=-b_{z^i}(\tau)\phi,$$

$$\frac{\partial^2\phi}{\partial^2\kappa^*}=\phi,\quad \frac{\partial^2\phi}{\partial^2 p^*}=2\,\frac{1}{p^2}\phi,\quad \frac{\partial^2\phi}{\partial v^2}=b_v^2(\tau)\phi,\quad \frac{\partial^2\phi}{\partial w^2}=b_w^2(\tau)\phi,\quad \frac{\partial^2\phi}{\partial^2 z^i}=b_{z^i}^2(\tau)\phi,$$

$$\frac{\partial^2\phi}{\partial k\partial p}=\frac{1}{p}\phi,\quad \frac{\partial^2\phi}{\partial k\partial v}=b_v(\tau)\phi,\quad \frac{\partial^2\phi}{\partial k\partial v}=b_v(\tau)\phi,\quad \frac{\partial^2\phi}{\partial k\partial z^i}=b_{z^i}(\tau)\phi,$$

$$\frac{\partial^2\phi}{\partial p\partial v}=\frac{1}{p}b_v(\tau)\phi,\quad \frac{\partial^2\phi}{\partial p\partial w}=\frac{1}{p}b_w(\tau)\phi,\quad \frac{\partial^2\phi}{\partial p\partial z^i}=\frac{1}{p}b_{z^i}(\tau)\phi,\quad \frac{\partial^2\phi}{\partial v\partial w}=b_v(\tau)b_w(\tau)\phi.$$

After substituting the variance and covariance terms and some algebraic manipulation (assuming that $\rho_{v,M}=0$), it is possible to obtain that the infinitesimal generator $\mathscr{A}$ is an affine function of underlying state variables $[v_t, w_t, z_t^i]$ of the form

$$\mathscr{A}\phi(\kappa_t,p_t,v_t,w_t\mathbf{z}_t;\tau)=-\left[\eta_0+\eta_v v_t+\eta_v w_t+\sum_{i=1}^{n}\eta_z z_t^i\right]\phi(\cdot) \tag{A.17}$$

where the parameters $\eta_0,\eta_v$, and $\eta_{z^i}$ are time invariant and depend only on the set of structural parameters of the model $\Omega$ and a set of functions $b_z(\tau)$ and $b_v(\tau)$ of the maturity of the bond:

$$\eta_0=A_0(\Omega)+A_z(\Omega)b_z(\tau)+A_v(\Omega)b_v(\tau)+A_w(\Omega)b_w(\tau)+B_v(\Omega)b_v^2(\tau)+B_w(\Omega)b_w^2(\tau),$$

$$\eta_j=\Theta_0^j(\Omega)+\Theta_1^j(\Omega)b_j(\tau)+\Theta_2^j(\Omega)b_j^2(\tau)\quad \text{for } j=v,w,z^i,$$

where the functions

$$\begin{aligned}&A_0(\Omega),A_j(\Omega),,B_w(\Omega)B_v(\Omega)\\&\quad\Theta_0^j(\Omega),\Theta_1^j(\Omega),\Theta_2^j(\Omega)\end{aligned} \tag{A.18}$$

are known nonlinear functions of the structural parameters $\Omega$ of the models as obtained from aggregating terms in (A.16). To save space we do not present the algebra for this step. The solution of the bond price is obtained by matching the coefficients of the state variables in (A.17) and (A.15). This gives rise to the following system of ODE:

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_z b_z(\tau) + A_v b_v(\tau) + B_v b_v^2(\tau) + A_w(\Omega) b_w(\tau) + B_w(\Omega) b_w^2(\tau), \quad \text{(A.19)}$$

$$-b_j'(\tau) = \Theta_0^j + \Theta_1^j b_j(\tau) + \Theta_2^j b_j^2(\tau) \quad \text{for } j = v, w, z^i. \quad \text{(A.20)}$$

The solution to this system of ODEs is presented in Lemma 1. The nominal price of a nominal zero-coupon bond $B_t^\tau$, with time to maturity $\tau$, is a log-linear function of the real productivity and nominal shocks $z_t^i$, $v_t$, and $w_t$. The closed-form solution is

$$B_t^\tau(v_t, w_t, \mathbf{z}_t) = A(\tau;\Omega)\exp\left[-b_v(\tau;\Omega)v_t - b_w(\tau;\Omega)w_t - \sum_{i=1}^{n} b_{z^i}(\tau;\Omega)z_t^i\right], \quad \text{(A.21)}$$

$$A(\tau;\Omega) = \exp(A_0(\Omega)\tau)a_v(\tau;\Omega)a_w(\tau;\Omega)c_v(\tau;\Omega)c_w(\tau;\Omega)\prod_{i=1}^{n} a_{z^i}(\tau;\Omega),$$

where $A(\tau;\Omega), b_v(\tau;\Omega)$, and $b_{z^i}(\tau;;\Omega)$ are obtained as solutions of (A.19)–(A.21). The general solution of this system of ODEs is given in the following lemma.

**Lemma** (*general solution of the pricing ODE*). *Consider the system of ODEs* (A.19) *and* (A.20). *It can be immediately noticed that the ordinary differential equations for both the nominal and real factors* (A.20) *have the same general form. Moreover, it can be verified by direct substitution that they admit a solution of the following form. For* $j = z^i, v, w$,

$$b^j(\tau) = \frac{1}{2\Theta_2^j}\left(-\Theta_1^j + \sqrt{D^j}\tan\left[\arctan\left(\frac{\Theta_1^j}{\sqrt{D^j}}\right) - \frac{1}{2}\tau\sqrt{D^j}\right]\right)$$
$$\text{where } D^j = -\Theta_1^{j2} + 4\Theta_0^j\Theta_2^j. \quad \text{(A.22)}$$

Let us now turn to solve equation (A.19):

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + A_w b_w(\tau) + B_v b_v^2(\tau) + B_w b_w^2(\tau) + \sum_{i=1}^{n} A_{z^i} b_{z^i}(\tau)$$

and let us consider the following educated guess: $A(\tau) = \exp(a_0\tau)a_v(\tau)c_v(\tau)a_w(\tau)c_w(\tau)\prod_{i=1}^{n} a_{z^i}(\tau)$. This implies

$$\frac{A'(\tau)}{A(\tau)} = \frac{\mathrm{d}}{\mathrm{d}\tau}[\ln A(\tau)] = a_0 + \frac{a_v'(\tau)}{a_v(\tau)} + \frac{a_w'(\tau)}{a_w(\tau)} + \frac{c_v'(\tau)}{c_v(\tau)} + \frac{c_w'(\tau)}{c_w(\tau)} + \sum_{i=1}^{n}\frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)}.$$

Thus, the pricing restriction is equivalent to

$$a_0 + \frac{a_v'(\tau)}{a_v(\tau)} + \frac{a_w'(\tau)}{a_w(\tau)} + \frac{c_v'(\tau)}{c_v(\tau)} + \frac{c_w'(\tau)}{c_w(\tau)} + \sum_{i=1}^{n} \frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)}$$
$$= A_0 + A_v b_v(\tau) + A_w b_w(\tau) + B_v b_v^2(\tau) + B_w b_w^2(\tau) + \sum_{i=1}^{n} A_{z^i} b_{z^i}(\tau)$$

which can be solved by solving the following system of individual restrictions:

$$a_0 = A_0; \quad \frac{a_v'(\tau)}{a_v(\tau)} = A_v b_v(\tau), \quad \frac{a_w'(\tau)}{a_w(\tau)} = A_w b_w(\tau),$$

$$\frac{c_v'(\tau)}{c_v(\tau)} = B_v b_v^2(\tau); \quad \frac{c_v'(\tau)}{c_v(\tau)} = B_w b_w^2(\tau); \quad \frac{a_{z^i}'(\tau)}{a_{z^i}(\tau)} = A_{z^i} b_{z^i}(\tau).$$

Let $j = v, z^i, w$. It can be verified by direct substitution that the solutions are

$$a_j(\tau) = 2^{A_j/\Theta_2^j} \exp\left(-\frac{A_j \tau \Theta_1^j}{2\Theta_2^j}\right) \left[\cos\left(\arctan\left(\frac{\Theta_1^j}{\sqrt{D^j}}\right) - \frac{1}{2}\tau\sqrt{D^j}\right)\right]^{A_j/\theta_2^j}$$
$$\times \left(\frac{\Theta_2^j \Theta_2^j}{D^j}\right)^{A_j/2\theta_2^j} \quad \text{where } D^j = -\Theta_1^{j2} + 4\Theta_0^j \Theta_2^j, \tag{A.23}$$

$$c_j(\tau) = \exp\left[\frac{1}{2\Theta_2^j}\left(B_j\Theta_2^j + B_j\tau\Theta_2^j - 2B_j\tau\Theta_2^j\Theta_2^j - B_j\Theta_1^j \log\left(1 + \frac{\Theta_2^j}{D^j}\right)\right.\right.$$
$$\left.\left. - B_j\sqrt{D^j} \tan\left[\arctan\left(\frac{\Theta_1^j}{\sqrt{D^j}}\right) - \frac{1}{2}\tau\sqrt{D^j}\right]\right)\right]$$
$$\times \left[\cos\left(\arctan\left(\frac{\Theta_1^j}{\sqrt{D^j}}\right) - \frac{1}{2}\tau\sqrt{D^j}\right)\right]^{-B_j\Theta_1^j/\Theta_2^{j2}}$$
$$\text{where } D^j \text{ is} - \Theta_1^{j2} + 4\Theta_0^j\Theta_2^j. \tag{A.24}$$

It is worth noticing that in order to avoid the existence of arbitrage, the previous solution should also be nonperiodic. Let us define $D \equiv -\Theta_1^2 + 4\Theta_0\Theta_2$. A well-known necessary and sufficient condition for the solution of a Riccati equation of the type $-b(\tau) = \Theta_0 + \Theta_1 b(\tau) + \Theta_2 b^2(\tau)$ to be nonperiodic is that $D < 0$.

### *A.3. The term structure of real interest rates*

From the standard first-order conditions of the representative agent, we know the real price of a zero-coupon inflation-linked (real) bond is equal, in equilibrium, to the conditional expected value of the product of the intertemporal marginal rate of substitution. If $IL_t^\tau$ is the price of an index-linked bond with $\tau$ years to maturity, then

$$IL_t^\tau = \mathrm{E}_t\left[\mathrm{e}^{-\rho\tau}\frac{\exp(-\ln X^*_{t+\tau})}{\exp(-\ln X^*_t)}\right] = \frac{1}{\exp(-\kappa^*_t)}\mathrm{E}_t[\exp(-\kappa^*_{t+\tau})]. \tag{A.25}$$

Let $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t; \tau)$ be the solution of the stochastic problem:

$$\phi(\kappa_t^*, p_t^*, v_t, w_t\mathbf{z}_t; \tau) = \mathrm{E}_t[\exp(-\kappa^*_{t+\tau})]. \tag{A.26}$$

Since the economy has a constant return to scale production process and logarithmic preferences, let us consider the following log-linear guess:

$$\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = [\exp(-\kappa_t^*)]A(\tau)\exp\left[-b_v(\tau)v_t - b_w(\tau)w_t - \sum_{i=1}^n b_{z^i}(\tau)z_t^i\right]. \tag{A.27}$$

If $2\theta \geqslant \sigma_v^2$, then zero is an unattainable boundary for $v_t$ and $p_t$ and $\phi(\kappa_t^*, p_t^*, v_t, w_t, z_t; \tau)$ satisfies assumption (A2) and the polynomial growth condition (A3) of Lemma 1. By the Yamada-Watanabe theorem, it is possible to show that on $R_+^2$ the vector diffusion process $[p_s, v_s]$ has a weak solution for any initial condition $[t; p_t, v_t] \in [0, T] \times R_{++}^2$ so that it satisfies condition (A1) of Lemma 1. $\phi(\kappa_t^*, p_t^*, v_t, w_t, z_t; \tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, from which we know that if $\phi(\kappa_t^*, p_t^*, v_t, w_t, z_t; \tau)$ is a solution to the stochastic problem (A.11), then it must also be a solution of the following differential problem:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \mathscr{A}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau), \tag{A.28}$$

$$\text{s.t.} \quad \lim_{\tau\to 0}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = [\exp(-\kappa_t^*)].$$

The left-hand side of (A.14) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\phi^* = \left[\frac{A'(\tau)}{A(\tau)} - b_v'(\tau)v_t - b_w'(\tau)w_t - \sum_{i=1}^n b_{z^i}'(\tau)z_t^i\right]\phi^*. \tag{A.29}$$

Applying Ito's Lemma to the right-hand side of (A.14), it is possible to obtain (after some algebraic manipulation) that the infinitesimal generator $\mathscr{A}$ is an affine function of underlying state variables $[v_t, w_t z_t^i]$ of the form

$$\mathscr{A}\phi = \eta_0^{IL} + \eta_v^{IL}v_t + \eta_w^{IL}w_t + \sum_{i=1}^n \eta_z^{IL}z_t^i, \tag{A.30}$$

where the parameters $\eta_0, \eta_v, \eta_w$, and $\eta_{z^i}$ are not time varying and depend on the set of structural parameters of the model $\Omega$ and set of functions $b_z(\tau)$, $b_w(\tau)$, and $b_v(\tau)$ of the maturity of the bond only. The exact functional form of $b_v(\tau), b_w(\tau)$ and $b_z(\tau)$ is presented in Section.

Equating left-hand and right-hand side of Eq. (A.28) we obtain coefficients for the following system of ODE:

$$\frac{A'(\tau)}{A(\tau)} = A_0 + A_v b_v(\tau) + A_z b_z(\tau) + B_v b_v^2(\tau) + A_w b_w(\tau) + B_w b_w^2(\tau), \tag{A.31}$$

$$-b_j'(\tau) = \Theta_0^j + \Theta_1^j b_j(\tau) + \Theta_2^j b_j^2(\tau) \quad \text{for } j = v, w, z^i. \tag{A.32}$$

Solving the system of ODEs (generic solution is given by in section) we obtain Eq. (A.33) which is the analog of Eq. (A.21) for the nominal term structure:

$$B_t^\tau(v_t, w_t, \mathbf{z}_t) = A^{IL}(\tau;\Omega)\exp\left[-b_v^{IL}(\tau;\Omega)v_t - b_w^{IL}(\tau;\Omega)w_t - \sum_{i=1}^{n} b_{z^i}^{IL}(\tau;\Omega)z_t^i\right], \tag{A.33}$$

$$A^{IL}(\tau;\Omega) = \exp(A_0^{IL}(\Omega)\tau)a_v^{IL}(\tau;\Omega)c_v^{IL}(\tau;\Omega)a_w^{IL}(\tau;\Omega)c_w^{IL}(\tau;\Omega)\prod_{i=1}^{n} a_{z^i}^{IL}(\tau;\Omega).$$

### *A.4. The inflation risk premium*

The risk premium on the inflation rate is equal to $COV_t\left[e^{-\rho\tau}\frac{\exp(-\kappa_{t+\tau}^*)}{\exp(-\kappa_t^*)};\frac{p_t^*}{p_{t+\tau}^*}\right] = B_t^\tau - IL_t^\tau \times E_t[\frac{p_t^*}{p_{t+\tau}^*}]$. Thus, the risk premium can be calculated as a difference between the price of the nominal bond and the price of the real bond adjusted for expected inflation. In yield terms, the risk premium can be expressed in terms of the difference between the yield on the nominal bond and the yield on the index-linked bond adjusted by inflation. Proposition 3 provides closed-form solutions for $B_t^\tau$, while $IL_t^\tau$ was previously computed. Thus, in order to obtain the closed-form solution of the inflation risk premium, we just need to compute the expected value of the reciprocal of the price process.

Let $\phi(p_t^*, \kappa_t^*, v_t, w_t, z_t;\tau)$ be the solution of the stochastic problem

$$\phi(p_t^*, \kappa_t^*, v_t, w_t\mathbf{z}_t;\tau) = E_t\left[\frac{1}{p_{t+\tau}^*}\right]. \tag{A.34}$$

Let us consider the following log-linear guess:

$$\phi(p_t^*, \kappa_t^*, v_t, w_t, \mathbf{z}_t;\tau) = \left[\frac{1}{p_t^*}\right]A(\tau)\exp\left[-b_v^p(\tau)v_t - b_w^p(\tau)w_t - \sum_{i=1}^{n} b_{z^i}^p(\tau)z_t^i\right]. \tag{A.35}$$

Since $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t;\tau)$ is bounded below and satisfies the regularity conditions of Lemma 1, we know that if $\phi(\kappa_t^*, p_t^*, v_t, w_t z_t;\tau)$ is a solution to the stochastic problem

(A.34), then it must also be a solution of the following differential problem:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\,\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = \mathscr{A}\phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau), \tag{A.36}$$

$$\text{s.t.}\quad \lim_{\tau\to 0}\, \phi(\kappa_t^*, p_t^*, v_t, w_t, \mathbf{z}_t; \tau) = [\exp(-\kappa_t^*)].$$

Using Lemma 1 and mimicking the approach used for the price of nominal and real bond prices, we can derive the functional form of the perameters $b_{\cdot}^p(\tau;\Omega)$. We obtain

$$\mathrm{E}_t\left[\frac{1}{p_{t+\tau}^*}\right] = \frac{1}{p_t^*}A^p(\tau;\Omega)\,\exp\left[-b_v^p(\tau;\Omega)v_t - b_w^p(\tau;\Omega)w_t - \sum_{i=1}^{n} b_{z^i}^p(\tau;\Omega)z_t^i\right], \tag{A.37}$$

$$A^p(\tau;\Omega) = \exp(A_0^p(\Omega)\tau)a_v^p(\tau;\Omega)c_v^p(\tau;\Omega)a_w^p(\tau;\Omega)c_w^p(\tau;\Omega)\prod_{i=1}^{n} a_{z^i}^p(\tau;\Omega).$$

The solution allows us to compute the inflation risk premium in closed form.

**Lemma 1.** *The second moments of the shifted square-root process $v_t$ is equal to*

$$\begin{aligned}\mathrm{E}_t(v_T^2) &= v_t^2\mathrm{e}^{2k(T-t)} + \left[\frac{1}{k}(\mathrm{e}^{2k(T-t)} - \mathrm{e}^{k(T-t)})\right](2\theta + \sigma_{1v}^2)\left(v_t + \frac{\theta}{k}\right)\\ &\quad + \left[\frac{1}{2k}(\mathrm{e}^{2k(T-t)} - 1)\right]\left[\sigma_{0v}^2 - (2\theta + \sigma_{1v}^2)\frac{\theta}{k}\right].\end{aligned}$$

*The conditional variance follows easily from $Var_t(v_T) = \mathrm{E}_t(v_T^2) - \mathrm{E}_t^2(v_T)$.*

**Proof.** Consider the canonical representation of the diffusion process of $d(v_t^2)$:

$$v_T^2 = v_t^2 + \int_t^T [2kv_u^2 + (2\theta + \sigma_{1v}^2)v_u + \sigma_{0v}^2]\,\mathrm{d}u + \int_t^T \sqrt{\sigma_{0v}^2 + \sigma_{1v}^2 v_u}\,\mathrm{d}W_u.$$

Taking the conditional expectation and using Fubini's theorem and the law of iterated expectations, we have $\mathrm{E}_t(v_T^2) = v_t^2 + \int_t^T [2k\mathrm{E}_t(v_u^2) + (2\theta + \sigma_{1v}^2)\mathrm{E}_t(v_u) + \sigma_{0v}^2]\,\mathrm{d}u$.

Taking the partial derivatives with respect to $T$, we get $\frac{\partial}{\partial T}\mathrm{E}_t(v_T^2) = 2k\mathrm{E}_t(v_T^2) + (2\theta + \sigma_{1v}^2)\mathrm{E}_t(v_T) + \sigma_{0v}^2$. Since $\mathrm{E}_t(v_T)$ is known, we can solve the previous ordinary differential equation with respect to $\mathrm{E}_t(v_T^2)$, subject to $\mathrm{E}_t(v_t^2) = v_t^2$. The result follows.

The conditional variance follows easily from $Var_t(v_T) = \mathrm{E}_t(v_T^2) - \mathrm{E}_t^2(v_T)$. The transition density of the nominal factor is a shifted Chi-square distribution. Define $f_t = \sigma_{0v}^2 + \sigma_{1v}^2 v_t$; the conditional distribution of $f_t$ is a noncentral Chi-square. Let $G(\overline{v}) = Pr(v_T < \overline{v}|v_t)$ denote the cumulative distribution of the original process. The nominal factor's distribution can be obtained from the noncentral $\chi^2$ by taking a linear transformation of its argument, $Pr\left(\frac{f_T - \sigma_{0v}^2}{\sigma_{1v}^2} < \overline{v}|f_t\right) = \chi^2(\sigma_{1v}^2\overline{v} + \sigma_{0v}^2)$. □

**Proposition 5.** *The linear regression coefficient* $\beta_{n,m}$ *implied by the structural model is equal to*

$$\beta_{n,m} = \frac{(n-m)}{m\phi(n,m)}\left\{\left[\frac{b_v(n-m)}{n-m}b_v^0(n,m)\right]\left(\frac{\sigma_{1v}^2\theta_v}{2k_v^2}-\frac{\sigma_{0v}^2}{2k_v}\right)\mathrm{e}^{km}\right.$$
$$-\left[\frac{b_v(n)}{n}b_v^0(n,m)\right]\left(\frac{\sigma_{1v}^2\theta_v}{2k_v^2}-\frac{\sigma_{0v}^2}{2k_v}\right)$$
$$\times\left[\frac{b_w(n-m)}{n-m}b_w^0(n,m)\right]\left(\frac{\sigma_{1w}^2\theta_w}{2k_w^2}-\frac{\sigma_{0w}^2}{2k_w}\right)\mathrm{e}^{k_w m}$$
$$-\left[\frac{b_w(n)}{n}b_w^0(n,m)\right]\left(\frac{\sigma_{1w}^2\theta_w}{2k_w^2}-\frac{\sigma_{0w}^2}{2k_w}\right)$$
$$\left.+\sum_{i=1}^{n}\left[\frac{b_{z^i}(n-m)}{n-m}b_{z^i}^0(n,m)\right]\frac{\sigma_z^2\zeta}{2\xi^2}\,\mathrm{e}^{\xi m}-\sum_{i=1}^{n}\left[\frac{b_{z^i}(n)}{n}b_{z^i}^0(n,m)\right]\frac{\sigma_z^2\zeta}{2\xi^2}\right\}.$$

**Proof.** The regression coefficient $\beta_{n,m}$ is equal to

$$\beta_{n,m} = \frac{(\frac{m}{n-m})cov(y_{t+m}^{n-m}-y_t^n, y_t^n-y_t^m)}{(\frac{m}{n-m})^2 var(y_t^n-y_t^m)}.$$

Define $a(n-m,n) = -\frac{1}{n-m}\ln A(n-m)+\frac{1}{n}\ln A(n)$, $b^0(n,m)=\frac{b(n)}{n}-\frac{b(m)}{m}$, and let

$$R_{t+m}^{n-m}-R_t^n = a(n-m,m)+\left[\frac{b_v(n-m)}{n-m}v_{t+m}-\frac{b_v(n)}{n}v_t\right]+\left[\frac{b_w(n-m)}{n-m}w_{t+m}-\frac{b_w(n)}{n}w_t\right]$$
$$+\sum_{i=1}^{n}\left[\frac{b_{z^i}(n-m)}{n-m}\,z_{t+m}^i-\frac{b_{z^i}(n)}{n}\,z_t^i\right].$$

The denominator of the regression coefficient is equal to

$$var(R_t^n-R_t^m) = [b_v^0(n,m)]^2\,Var(v_t)+[b_w^0(n,m)]^2\,Var(w_t)+\sum_{i=1}^{n}[b_{z^i}^0(n,m)]^2\,Var(z_t^i)$$
$$\equiv \phi(n,m).$$

The numerator of the regression coefficient is equal to

$$cov(R_{t+m}^{n-m}-R_t^n, R_t^n-R_t^m) = \left[\frac{b_v(n-m)}{n-m}\,b_v^0(n,m)\right]cov(v_{t+m},v_t)$$
$$-\left[\frac{b_v(n)}{n}b_v^0(n,m)\right]var(v_t)$$
$$+\left[\frac{b_w(n-m)}{n-m}b_w^0(n,m)\right]cov(w_{t+m},w_t)$$

$$- \left[ \frac{b_w(n)}{n} b_w^0(n,m) \right] var(w_t)$$
$$+ \sum_{i=1}^{n} \left[ \frac{b_{z^i}(n-m)}{n-m} b_{z^i}^0(n,m) \right] cov(z_{t+m}^i, z_t^i)$$
$$- \sum_{i=1}^{n} \left[ \frac{b_{z^i}(n)}{n} b_{z^i}^0(n,m) \right] var(z_t^i).$$

Lemma 1 gives closed-form solutions for the second moments of the factors. $\mathrm{E}(v_t) = -\frac{\theta}{k}$, $\mathrm{E}(z_t) = -\frac{\zeta}{\xi}$, $Var(v_t) = \frac{\sigma_{1v}^2\theta}{2k^2} - \frac{\sigma_{0v}^2}{2k}$, $Var(z_t) = \sigma_z^2 \frac{\zeta}{2\xi^2}$, $\mathrm{E}(v_t^2) = \frac{\sigma_{1v}^2\theta}{2k^2} - \frac{\sigma_{0v}^2}{2k} + \frac{\theta^2}{k^2}$, and $\mathrm{E}(z_t^2) = \sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2}$. Similarly, the two covariance terms can be calculated as

$$\begin{aligned}
\mathrm{E}[z_{t+m}z_t] &= \mathrm{E}[\mathrm{E}_t(z_{t+m})z_t] = \mathrm{E}\left( \left[ -\frac{\zeta}{\xi} + \mathrm{e}^{\xi m}\left( z_t + \frac{\zeta}{\xi} \right) \right] z_t \right) \\
&= \mathrm{e}^{\xi m}\left[ \sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2} \right] - \left[ \frac{\zeta}{\xi}(\mathrm{e}^{\xi m} - 1) \right] \frac{\zeta}{\xi}, \\
cov(z_{t+m}, z_t) &= \mathrm{E}[z_{t+m}z_t] - [\mathrm{E}(z_t)]^2 \\
&= \mathrm{e}^{\xi m}\left[ \sigma_z^2 \frac{\zeta}{2\xi^2} + \frac{\zeta^2}{\xi^2} \right] - \left[ \frac{\zeta}{\xi}(\mathrm{e}^{\xi m} - 1) \right] \frac{\zeta}{\xi} - \frac{\zeta^2}{\xi^2} = \mathrm{e}^{\xi m}\sigma_z^2 \frac{\zeta}{2\xi^2}.
\end{aligned}$$

Similarly, for the nominal factor we obtain

$$\begin{aligned}
cov(v_{t+m}, v_t) &= \mathrm{E}[v_{t+m} - \mathrm{E}(v_{t+m})(v_t - \mathrm{E}(v_t))] = \mathrm{E}[v_{t+m}v_t] - [\mathrm{E}(v_t)]^2 \\
&= \mathrm{e}^{km}\left( \frac{\sigma_{1v}^2\theta}{2k^2} - \frac{\sigma_{0v}^2}{2k} \right).
\end{aligned}$$

The result follows. □

## References

Andrews, D.W.K., Ploberger, W., 1994. Optimal tests when a nuisance parameter is present only under the alternative. Econometrica 62, 1383–1414.

Ang, A., Bekaert, G., 2004. The term structure of real rates and expected inflation. Unpublished working paper, Columbia University.

Backus, D., Telmer, C., Wu, L., 1999. Design and estimation of affine yield models. Unpublished working paper, Carnegie Mellon University.

Bakshi, G.S., Chen, Z., 1996. Inflation, asset prices, and the term structure of interest rates in monetary economies. The Review of Financial Studies 9 (1), 241–275.

Bansal, R., Zhou, H., 2002. Term structure of interest rates with regime shifts. Journal of Finance 57, 1997–2043.

Bekaert, G., Hodrick, R.J., Marshall, D., 1997. On biases in tests of the expectation hypothesis of the term structure of interest rates. Journal of Financial Economics 44, 309–348.

Benhabib, J., Schmitt-Grohé, S., Uribe, M., 2001. Monetary policy and multiple equilibria. American Economic Review 91 (1), 167–186.

Benhabib, J., Schmitt-Grohé, S., Uribe, M., 1999. The perils of Taylor rules. CEPR Discussion Papers 2314.

Bernanke, B., Gertler, M., 1989. Agency costs, net worth, and business fluctuations. American Economic Review 79 (1), 14–31.

Bernanke, B., Gertler, M., Gilchrist, S., 2000. The financial accelerator in a quantitative business cycle framework. In: Taylor, J., Woodford, M. (Eds.), Handbook of Macroeconomics, vol. 1C. Elsevier, Amsterdam, pp. 1341–1393.

Benninga, S., Protopapadakis, A., 1983. Real and nominal interest rates under uncertainty: the Fisher theorem and the term structure. Journal of Political Economy 91, 856–867.

Blanchard, O.J., Kiyotaki, N., 1987. Monopolistic competition and the effects of aggregate demand. American Economic Review 77 (4), 647–666.

Boudoukh, J., 1993. An equilibrium model of nominal bond prices with inflation-output correlation and stochastic volatility. Journal of Money, Credit and Banking 25, 636–665.

Campbell, J.Y., Shiller, R.J., 1991. Yield spreads and interest rates movements, a bird's eye view. The Review of Economic Studies 58, 495–514.

Carlstrom, C.T., Fuerst, T.S., 1998. Monetary shocks, agency costs, and the business cycles. Economic Theory 12 (3), 599–620.

Chan, K.C., Karolyi, G.A., Longstaff, F., Sanders, A., 1992. An empirical comparison of alternative models of the short-term interest rates. Journal of Finance 47 (3), 1209–1227.

Chari, V.V., Kehoe, P.J., McGrattan, E.R., 1996. Sticky price models of the business cycle: can the contract multiplier solve the persistence problem? Econometrica 68 (5), 1151–1179.

Christiano, L.J., Eichenbaum, M., Evans, C., 2001. Nominal rigidities and the dynamic effects of a shock to monetary policy. NBER Working Paper No. w8403.

Christiano, L.J., Rostagno, M., 2001. Money growth monitoring and the Taylor rule, NBER Working Paper 8539.

Chen, R., Scott, L., 1993. Maximum likelihood estimation for a multifactor equilibrium model of the term structure of interest rates. Journal of Fixed Income 3 (3), 14–31.

Cochrane, J., Piazzesi, M., 2002. Bond Risk Premia, NBER Working Paper 9178.

Constantinides, G.M., 1992. A theory of the nominal term structure of interest rates. Review of Financial Studies 5 (4), 531–552.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1981. The relation between forward prices and futures prices. Journal of Financial Economics 9 (4), 321–346.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1985a. An intertemporal general equilibrium model of asset prices. Econometrica 53, 363–383.

Cox, J.C., Ingersoll, J.E., Ross, S.A., 1985b. A theory of the term structure of interest rates. Econometrica 53, 385–407.

Cooley, T.F., Hansen, G.D., 1995. Money and the business cycle. In: Cooley, T., Prescott, E. (Eds.), Frontiers of Business Cycle Research. Princeton University Press, Princeton, NJ.

Cooley, T.F., Nam, K., 1998. Asymmetric information, financial intermediation, and business cycles. Economic Theory 12, 599–620.

Dai, Q., Singleton, K., 2000. Specification analysis of affine term structure modes. Journal of Finance 55, 1943–1978.

Dai, Q., Singleton, K., 2002. Expectations puzzles, time-varying risk premium, and affine models of the term structure. Journal of Financial Economics 63, 415–441.

Danthine, J.P., Donadson, J., Smith, L., 1987. Superneutrality of money in a stochastic dynamic macroeconomic model. Journal of Monetary Economics 20, 475–499.

Duarte, J., 2000. The relevance of the price of risk for affine term structure models, Unpublished working paper, The University of Chicago.

Duffie, D., Kan, R., 1996. A yield factor model of interest rates. Mathematical Finance 6 (4), 379–406.

Duffee, R.G., Stanton, R.H., 2001. Estimation of dynamic term structure models. Unpublished working paper, Haas School of Business, University of California Berkeley.

Duffee, G., 2002. Term premia and interest rate forecasts in affine models. The Journal of Finance 57, 405–443.

Erceg, C.J., 1997. Nominal wage rigidities and the propagation of monetary disturbances. Board of Governors of the Federal Reserve System, International Finance Discussion Papers 590.

Erceg, C.J., Henderson, D.W., Dale, W., Levin, A.T., 2000. Optimal monetary policy with staggered wage and price contracts. Journal of Monetary Economics 46 (2), 281–313.

Evans, M.D., 1998. Real rates, expected inflation, and inflation risk premia. The Journal of Finance 53 (1), 187–218.

Evans, M.D., 2003. Real risk, inflation risk and the term structure. The Economic Journal 114, 345–389.

Fama, E.F., 1976a. Forward rates as predictors of future spot rates. Journal of Financial Economics 3, 361–377.

Fama, E.F., 1976b. Inflation uncertainty and expected returns on Treasury bills. Journal of Political Economy, 427–448.

Fama, E.F., 1979. Stock returns, real activity, inflation and money. American Economic Review 71, 545–565.

Fama, E.F., 1990. Term structure forecasts of interest rates, inflation, and real returns. Journal of Monetary Economics 25 (1), 59–76.

Fama, E.F., Bliss, R., 1987. The information in long maturity forward rates. American Economic Review 77, 680–692.

Fama, E.F., Gibbons, M.R., 1982. Inflation, real returns and capital investment. Journal of Monetary Economics 9, 297–323.

Feldstein, M., 1980. Inflation, tax rules and the stock market. Journal of Monetary Economics 6, 309–331.

Feldstein, M., Green, J., Sheshinski, E., 1978. Inflation and taxes in a growing economy with debt and equity finance. Journal of Political Economy 86 (2), S53–S70.

Fisher, M., Gilles, C., 1996. Estimating exponential-affine models of the term structure. Unpublished working paper, Federal Reserve Board, Washington, DC.

Fisher, S., Modigliani, F., 1978. Towards an understanding of the real effects and costs of inflation. Weltwirtshaftliches Archiv, 810–832.

Fisher, J.D.M., 1999. Credit market imperfections and the heterogeneous response of firms to monetary shocks. Journal of Money, Credit and Banking 31, 187–211.

Fuerst, T.S., 1995. Monetary and financial interactions in the business cycle. Journal of Money, Credit and Banking 27 (4), 1321–1338.

Gali, J., Lopez-Salido, D., Valles, J., 2002. Technology shocks and monetary policy: assessing the Fed's performance. NBER Working Papers 8768, National Bureau of Economic Research, Inc., 1–29.

Gibbons, M.R., Ramaswamy, K., 1993. A test of the Cox, Ingersoll and Ross model. Review of Financial Studies 6, 619–658.

Graham, J.R., 1996. Proxies for the corporate marginal tax rate. Journal of Financial Economics 42, 187–221.

Green, R., Hollifield, B., 2003. The personal-tax advantages of equity. Journal of Financial Economics, 67, 175–216.

Hansen, B., 1996. Inference when a nuisance parameter is not identified under the null hypothesis. Econometrica 64, 413–430.

Harvey, C.R., 1988. The real term structure and consumption growth. Journal of Financial Economics 22, 305–334.

Huang, K.X.D., Liu, Z., 1999. Staggered contracts and business cycle persistence. Discussion Paper 127, Federal Reserve Bank of Minneapolis, Institute for Empirical Macroeconomics.

Ireland, P., 2001. Sticky-prices models of the business cycle: specification and stability. Journal of Monetary Economics 47, 3–18.

Kwon, 1992. The time variant term premium of interest rates. Ph.D. Dissertation, Ohio State University, Economics Department.

Levin, A.T., Wieland, V., Williams, J.C., 1999. Robustness of simple monetary policies under model uncertainty. In: Taylor, J. (Ed.), Monetary Policy Rules. University of Chicago Press, Chicago, pp. 263–299.

Litterman, R., Scheinkman, J., 1991. Common factors affecting bond returns. Journal of Fixed Income 1 (1), 54–61.

Longstaff, F.A., Schwartz, E.S., 1992. Interest rate volatility and the term structure: a two factor general equilibrium model. Journal of Finance 47 (4), 1259–1282.

Marshall, D., 1992. Inflation and asset returns in a monetary economy. Journal of Finance 47 (4), 1315–1342.

DeAngelo, H., Masulis, R.W., 1980. Optimal capital structure under corporate and personal taxation. Journal of Financial Economics 8, 3–29.

McCulloch, H., 1975. The tax-adjusted yield curve. Journal of Finance 30 (3), 811–830.

McCulloch, H., 1990. U.S. Term structure data, 1946–87. in: Handbook of Monetary Economics, vol. I, pp. 672–715.

McCulloch, H., 1993. A reexamination of traditional hypotheses about the term structure: a comment. Journal of Finance 48 (2), 79–89.

McIntyre, R.S., Coo Nguyen, T.D., 2000. Corporate income taxes in the 1990s. Institute of Taxation and Economic Policy, Washington, DC.

Naik, V., Lee, M.H., 1997. Yield curve dynamics with discrete shifts in economic regimes, theory and estimation. Unpublished working paper, Faculty of Commerce, University of British Columbia.

Orphanides, A., 2003. The quest for prosperity without inflation. Journal of Monetary Economics 50 (3), 633–663.

Pearson, N.D., Sun, T.S., 1994. Exploiting the conditional density in estimating the term structure: an application of the Cox, Ingersoll and Ross model. Journal of Finance 49 (4), 1279–1304.

Pennacchi, G.G., 1991. Identifying the dynamics of real interest rates and inflation: evidence using survey data. Review of Financial Studies 4 (1), 53–86.

Piazzesi, M., 2004, Bond yields and the Federal Reserve. Journal of Political Economy, forthcoming.

Rotemberg, J.J., Woodford M., 1999. Interest-rate rules in an estimated sticky price model. NBER Working Papers 6618, National Bureau of Economic Research.

Rudenbush, G.D., Svensson, L.E., 1999. Policy rules for inflation targeting. In: Taylor, J. (Ed.), Monetary Policy Rules. University of Chicago Press, Chicago, pp. 203–253.

Stambaugh, R.F., 1988. The information in forward rates: implications for models of the term structure. Journal of Financial Economics 21, 41–70.

Stokey, N.L., Lucas Jr., R.E., 1989. Recursive methods in economic dynamics. Harvard University Press.

Taylor, J.B., 1993. Discretion versus policy rules in practice. Carnegie-Rochester Conference Series on Public Policy 39, pp. 195–214.

Vasiček, O., 1977. An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177–188.